2004 Missouri Collegiate Mathematics Competition

Session I

1. Let $P \neq (0,0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at P will intersect the x-axis at a point, say Q. Let $O = (0,0)$ and form the triangle OPQ. Let OQ be the base of this triangle. Find the minimum ratio of the length of the base of $\triangle OPQ$ to its height.

Solution.

Let $P = (p, p^2)$. The slope of the tangent line to the parabola at P is $2p$, so the slope of the normal line to the parabola at P is $-1/2p$. Thus, the equation of the normal line to the parabola at P is

$$y - p^2 = -\frac{1}{2p}(x - p), \;\; p \neq 0.$$

The point Q lies on the normal line and has a y-coordinate of 0. Therefore, the x-coordinate of Q is $p + 2p^3$. Hence, the length of the base of $\triangle OPQ$ is $|p + 2p^3|$ and the height is p^2. Therefore, the ratio $R(p)$ of the length of the base of this triangle to its height is

$$R(p) = \frac{|p + 2p^3|}{p^2} = \left|2p + \frac{1}{p}\right|.$$

Critical points occur where

$$R'(p) = \begin{cases} 2 - \frac{1}{p^2}, & \text{for } p > 0 \\ -2 + \frac{1}{p^2}, & \text{for } p < 0. \end{cases}$$

is zero; these occur at

$$p = \pm\frac{1}{\sqrt{2}}.$$

Thus, the minimum ratio is

$$R_{\min}(p) = \left| \pm \frac{2}{\sqrt{2}} \pm \sqrt{2} \right| = 2\sqrt{2}.$$

That this is a relative minimum of $R(p)$ follows because

$$R''(p) = \begin{cases} \frac{2}{p^3}, & \text{for } p > 0 \\ \frac{-2}{p^3}, & \text{for } p < 0 \end{cases}$$

is positive for any $p \neq 0$.

2. The numbers $\pm 1, \pm 2, \ldots \pm 2004$ are written on a blackboard. You decide to pick two numbers x and y at random, erase them, and write their product, xy, on the board. You continue this process until only one number remains. Prove that the last number is positive.

Solution.

Let

$$P = (1 \cdot 2 \cdot 3 \cdots 2004) \cdot (-1 \cdot -2 \cdot -3 \cdots - 2004).$$

P is obviously positive. If two numbrs x, y are removed from the list and the product xy is added to the list, then the product of the elements of the new list is still P. This could continue until only one element is left, and this element must necessarily be P which is positive.

From *Quantum,* "Problems Teach Us How to Think," 11.3 (Jan/Feb 2001), p. 43 by V. Proizvolov.

3. A chess position possesses the following property: On every vertical column and on every horizontal row, there is an odd number of pieces. Prove that there is an even number of pieces on black squares.

Solution.

A chessboard has a black square in the lower left corner. Let $1 \leq i \leq 8$. Let c_i denote the number of pieces in column i and r_i denote the number of pieces in row i. By the statement of the problem, every c_i and r_i is odd. The quantity

$$c_2 + c_4 + c_6 + c_8$$

is the number of pieces in columns 2, 4, 6, and 8. The quantity

$$r_1 + r_3 + r_5 + r_7$$

is the number of pieces in rows 1, 3, 5, and 7. Consider the sum

$$S = c_2 + c_4 + c_6 + c_8 + r_1 + r_3 + r_5 + r_7.$$

Each even-indexed c_k tallies all the black pieces in that column and all white pieces in an odd row and that even column. Each odd-indexed r_k tallies all the black pieces in that row and all white pieces in an even column and that odd row. All the black pieces on the board are tallied, and none is counted twice; let their number be S_1. The white pieces tallied by the c_k's are the same as the white pieces tallied by the r_k's; let their number be S_2. It follows that $S = S_1 + 2S_2$. Since S is even because it is the sum of 8 odd integers and $2S_2$ is even, then S_1 (the number of black pieces) must also be even.

4. At a point P on the curve

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1,$$

the tangent to the curve meets the x-axis at $(h, 0)$ and the y-axis at $(0, k)$. As P moves on the given curve, find the locus of points $Q(h, k)$.

Solution.

The algebraic work is easier if the equation of the curve is written in a parametric form

$$x = a\cos^3\theta \quad \text{and} \quad y = b\sin^3\theta.$$

Then

$$\frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta), \quad \frac{dy}{d\theta} = 3b\sin^2\theta(\cos\theta), \quad \text{and} \quad \frac{dy}{dx} = -\frac{b}{a}\tan\theta.$$

The equation of the tangent line at any point θ is given by

$$y - b\sin^3\theta = -\frac{b}{a}\tan\theta(x - a\cos^3\theta).$$

Then h and k, the x- and y- intercepts of this line, are

$$h = a\cos\theta \quad \text{and} \quad k = b\sin\theta.$$

This shows that the locus of the point $Q(h,k)$ is an ellipse given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

5. Let a, b, c, and d be integers. Suppose that each of the three quadratics ax^2+bx+c, $ax^2+bx+(c+d)$, and $ax^2+bx+(c+2d)$ factors over the integers, i.e. has rational roots. Let $S = ad > 0$. Show that S represents the area of some Pythagorean triangle (integer-sided right triangle).

Solution.

If ax^2+bx+c is factorable over the integers, then from the quadratic formula, b^2-4ac is a perfect square. Similarly, $b^2-4a(c+d)$ and $b^2-4a(c+2d)$ are also perfect squares. Assume $ad > 0$. Then there are positive integers $M > N > P$ such that

$$\begin{cases} M^2 = b^2 - 4ac, \\ N^2 = b^2 - 4a(c+d), \\ P^2 = b^2 - 4a(c+2d). \end{cases}$$

Hence, $M^2 - N^2 = N^2 - P^2 = 4ad$ and this implies that M, N, and P are either all even or all odd, and that $M^2 + P^2 = 2N^2$. This last equation can be rewritten as

$$\left(\frac{M-P}{2}\right)^2 + \left(\frac{M+P}{2}\right)^2 = N^2.$$

$(M-P)/2$ and $(M+P)/2$ are positive integers, so

$$\left(\frac{M-P}{2}, \frac{M+P}{2}, N\right)$$

is a Pythagorean triple, and thus the sides of a Pythagorean triangle whose area is

$$A = \frac{1}{2}\cdot\frac{M-P}{2}\cdot\frac{M+P}{2} = \frac{M^2-P^2}{8} = \frac{8ad}{8} = ad.$$

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Session II

From Problem 1.5.2, *Berkeley Problems in Mathematics*, Springer, 1998, QA43.S695.

1. Suppose f is a continuous real-valued function on the interval $[0,1]$. Show that

$$\int_0^1 x^2 f(x)\,dx = \frac{1}{3} f(\xi)$$

for some $\xi \in [0,1]$.

Solution.

Because f is continuous, it attains its minimum and maximum at points a and b, both in $[0,1]$, giving

$$f(a)\int_0^1 x^2 dx \le \int_0^1 x^2 f(x)\,dx \le f(b)\int_0^1 x^2\,dx$$

or

$$f(a) \le 3\int_0^1 x^2 f(x)\,dx \le f(b).$$

Thus, the Intermediate Value Theorem guarantees a point $\xi \in [0,1]$ such that

$$f(\xi) = 3\int_0^1 x^2 f(x)\,dx.$$

From *Crux Mathematicorum*, Problem 2384, November, 1998. Proposed by Paul Bracken, CRM, Université de Montréal, Québec. Solution by Michel Bataille, Rouen, France.

2. Prove that $2(3n-1)^n \ge (3n+1)^n$ for all nonnegative integers n.

Solution.

The inequality is obvious for $n = 0$, so we may assume $n \ge 1$. We have to prove

$$\left(\frac{3n-1}{3n+1}\right)^n \ge \frac{1}{2}, \tag{1}$$

or, equivalently,

$$n \ln \frac{3n-1}{3n+1} \ge \ln \frac{1}{2}. \tag{2}$$

To this aim, we introduce the function

$$f(x) = x \ln \frac{3x-1}{3x+1}$$

defined on $[1, \infty)$. We compute

$$f'(x) = \ln \frac{3x-1}{3x+1} + \frac{6x}{9x^2-1} \quad \text{and} \quad f''(x) = \frac{-12}{(9x^2-1)^2}.$$

Since $f''(x) < 0$, $f'(x)$ is strictly decreasing on $[1, \infty)$. Moreover,

$$\lim_{x \to \infty} f'(x) = 0,$$

so $f'(x) > 0$ for all $x \in [1, \infty)$. Hence f is increasing on $[1, \infty)$ and, since $f(1) = \ln(1/2)$, the inequality (2) follows.

Comment.

A natural tool for proving inequality (1) might have been Bernoulli's Inequality. However, this Inequality does not provide sharp enough lower bounds and is ineffective in the present problem.

3. Equations of two ellipses E_1 and E_2 are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2x}{c} = 0 \text{ and } \frac{x^2}{b^2} + \frac{y^2}{a^2} + \frac{2x}{c} = 0,$$

respectively. AB is a common tangent, meeting E_1 at A and E_2 at B. Prove that when A and B are joined to the origin O, angle AOB is a right angle.

Solution.

Since

$$\frac{x^2}{a^2} - \frac{2x}{c} + \frac{y^2}{b^2} = 0,$$

we have, by completing the square and some algebraic manipulations, that

$$\frac{x^2 - \frac{2a^2}{c}x + \frac{a^4}{c^2}}{a^2} + \frac{y^2}{b^2} = \frac{a^2}{c^2}$$

$$\frac{\left(x - \frac{a^2}{c}\right)^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2}{c^2}$$

$$\frac{\left(x - \frac{a^2}{c}\right)^2}{\frac{a^4}{c^2}} + \frac{y^2}{\frac{a^2b^2}{c^2}} = 1.$$

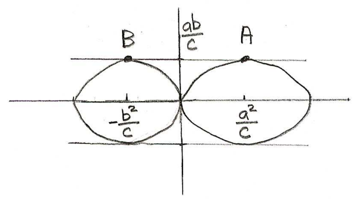
Similarly, since

$$\frac{x^2}{b^2} + \frac{2x}{c} + \frac{y^2}{a^2} = 0,$$

it follows that

$$\frac{\left(x - \frac{b^2}{c}\right)^2}{\frac{b^4}{c^2}} + \frac{y^2}{\frac{a^2b^2}{c^2}} = 1.$$

Graphing ellipses E_1 and E_2 we have the following figure.



and the common tangents are horizontal lines. Thus,

$$A = \left(\frac{a^2}{c}, \frac{ab}{c}\right) \quad \text{and} \quad B = \left(-\frac{b^2}{c}, \frac{ab}{c}\right).$$

Finally, the slopes of OA and OB are

$$\frac{ab}{a^2} = \frac{b}{a} \quad \text{and} \quad \frac{ba}{-b^2} = -\frac{a}{b},$$

respectively. Therefore $OA \perp OB$ so angle AOB is a right angle.

From the Ninth Irish Mathematical Olympiad (1996).

4. For each positive integer n, let $s(n)$ denote the sum of the digits of n (when n is written in base 10). Prove that for every positive integer n

$$s(2n) \leq 2s(n) \leq 10s(2n).$$

Solution.

The strict inequality $s(2n) < 10s(2n)$ is patently obvious. Now let $5, 6, 7, 8, 9$ be designated "big digits." If I is a big digit in the integer n, then it contributes $2s(I) = 2I$ to the quantity $2s(n)$. But

$$2I = 1 \cdot 10^1 + (2I - 10) \cdot 10^0,$$

so $s(2I)$ contributes

$$1 + (2I - 10) = 2I - 9$$

to the quantity $s(2n)$. Hence, if n contains L big digits, we have

$$s(2n) = 2s(n) - 9L,$$

that is, $s(2n) \leq 2s(n)$. Finally,

$$s(n) \geq \text{ sum of its big digits } \geq 5L,$$

so

$$20s(n) \geq 2s(n) + 90L,$$

or

$$20s(n) - 90L = 10s(2n) \geq 2s(n),$$

as desired.

5. The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, if $n \geq 2$. Use the Fibonacci numbers to express the number K_n of n-tuples $(x_1, x_2, \dots, x_n)$ of 0's, 1's, and 2's such that 0 is never followed by 1.

Solution.

$K_1 = 3 = F_4$ and $K_2 = 8 = F_6$. We shall prove $K_n = F_{2n+2}$ by induction. Assume $K_p = F_{2p+2}$ for $1 \leq p \leq m$. We are to show $K_{m+1} = F_{2(m+1)+2}$.

If a sequence of length $m+1$ starts with 1 or 2 then there are K_m subsequences of length m are available to follow either of them.

However, for a sequence starting with 0, the remaining subsequences of length m can only start with 0 or 2. Again, after the 2 there are K_{m-1} subsequences available. And after 0 only sequences sarting with either 0 or 2 are available. For the last one number of the sequence, after 0 it can only be either 0 or 2 and after 2 it has K_1 choices. Therefore,

$$\begin{aligned}
K_{m+1} &= K_m + K_m + K_{m-1} + K_{m-2} + \cdots + K_2 + K_1 + 1 + 1 \\
&= F_{2m+2} + F_{2m+2} + F_{2(m-1)+2} + F_{2(m-2)+2} + \cdots + F_6 + F_4 + F_2 + F_1 \\
\\
&= F_{2m+2} + F_{2m+2} + F_{2m} + F_{2m-2} + \cdots + F_6 + F_4 + F_3 \\
&= F_{2m+2} + F_{2m+2} + F_{2m} + F_{2m-2} + \cdots + F_6 + F_5 \\
&= F_{2m+2} + F_{2m+2} + F_{2m} + F_{2m-2} + \cdots + F_7 \\
&\dots \\
&= F_{2m+2} + F_{2m+2} + F_{2m} + F_{2m-1} \\
&= F_{2m+2} + F_{2m+2} + F_{2m+1} \\
&= F_{2m+2} + F_{2m+3} \\
&= F_{2m+4} \\
&= F_{2(m+1)+2}.
\end{aligned}$$

And this completes the induction.