2003 Missouri Collegiate Mathematics Competition

Session I

1. Let $P \neq (0,0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at P will intersect the parabola at another point, say Q. Find the coordinates of P so that the sum of the y-coordinates (or the sum of the ordinates) of P and Q is a minimum.

Solution.

Let $p > 0$, $P = (p, p^2)$ and $Q = (q, q^2)$. The slope of the tangent line to the parabola at P is $2p$, so the slope of the normal line to the parabola at P is $-1/2p$. Thus, the equation of the normal line to the parabola at P is

$$y - p^2 = -\frac{1}{2p}(x - p).$$

Since $Q = (q, q^2)$ lies on the normal line,

$$q^2 - p^2 = -\frac{1}{2p}(q - p) \quad \text{or} \quad q + p = -\frac{1}{2p}.$$

Solving for q, we obtain

$$q = -p - \frac{1}{2p}.$$

Now the sum of the y-coordinates of P and Q is

$$S = p^2 + q^2 = p^2 + \left(-p - \frac{1}{2p}\right)^2$$

$$= p^2 + p^2 + 1 + \frac{1}{4p^2} = 2p^2 + 1 + \frac{1}{4}p^{-2}.$$

Differentiating this quantity with respect to p, we obtain

$$\frac{dS}{dp} = 4p - \frac{1}{2}p^{-3} = \frac{1}{2}p^{-3}(8p^4 - 1).$$

Setting this quantity equal to 0 and solving for p, we have

$$p = \frac{1}{\sqrt[4]{8}}.$$

Therefore, the coordinates of P so that the sum of the y-coordinates of P and Q is a minimum are

$$\left(\frac{1}{\sqrt[4]{8}}, \frac{1}{\sqrt[2]{8}}\right).$$

2. There is an $m \times n$ rectangular array of office mailboxes in which mail for $p \leq mn$ people is distributed. Initially, the mailboxes are assigned alphabetically beginning at the upper left and proceeding down each column (the "next" mailbox to one at the bottom of a column is the one at the top of the next column to the right). A new secretary is hired, and decides that the mailboxes will now be assigned alphabetically beginning at the upper left and proceeding to the right across each row. Discuss, as completely as possible, whose mailboxes will be unchanged.

Solution.

The people in mailboxes $(1,1)$ and (m,n) (if $p = mn$) never change mailboxes. Consider mailbox (x,y). By simple counting using the two arrangements, (x,y) will not change if and only if

$$m(y-1) + x = n(x-1) + y$$

or

$$(n-1)(x-1) = (m-1)(y-1).$$

This is a linear equation in x and y whose graph is the diagonal containing both $(1,1)$ and (m,n) (note the positive y-axis points down with the given setup). Any point on this diagonal between $(1,1)$ and (m,n) with integer coordinates is a solution. If $m-1$ and $n-1$ are relatively prime, there are no such points, if $m-1$ and $n-1$ are not relatively prime there is a solution for each common divisor, and if $m = n$ every mailbox on the diagonal is unchanged. If $k > 1$ is the smallest nontrivial common divisor of $m-1$ and $n-1$, then the penultimate solution is

$$\left(\frac{m-1}{k} + 1, \frac{n-1}{k} + 1\right),$$

and this solution counts if

$$p \geq \frac{mn + k - 1}{k}.$$

In fact, for any common divisor d, the corresponding solution is counted whenever

$$p \geq \frac{mn + d - 1}{d}.$$

3. For sufficiently small but positive θ, the relation $\tan\theta > \theta$ holds. Prove, in the other direction, that for $0 < \theta < \pi/4$ one has

$$\tan\theta < \frac{4\theta}{\pi}.$$

Solution I.

Let $f(x) = \tan x$ and let $x = \theta$ be an arbitrary number in $(0, \pi/4)$. Then from the Mean Value Theorem (since $f(x)$ is continuous on $[0, \pi/4]$ and differentiable on $(0, \pi/4)$)

$$\begin{cases} \tan\theta = \tan 0 + \theta \sec^2\xi \\ \tan\pi/4 = \tan\theta + (\frac{\pi}{4} - \theta)\sec^2\xi' \end{cases},$$

where $0 < \xi < \theta$ and $\theta < \xi' < \pi/4$. But on the interval $(0, \pi/4)$ $\sec^2 x$ is an increasing function, so $\sec^2\xi' > \sec^2\xi$. This yields

$$\frac{1 - \tan\theta}{\frac{\pi}{4} - \theta} > \frac{\tan\theta}{\theta},$$

and algebraic rearrangement results in

$$\frac{4\theta}{\pi} > \tan\theta.$$

Solution II.

Let

$$f(\theta) = \tan\theta - \frac{4\theta}{\pi} \text{ for } \theta \in [0, \pi/4].$$

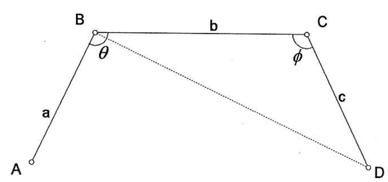
Then,

$$f'(\theta) = \sec^2\theta - \frac{4}{\pi}.$$

Therefore, there exists a unique $\xi \in (0, \pi/4)$ such that $f'(\xi) = 0$. Also, $f'(\theta) < 0$ for $\theta \in (0, \xi)$ and $f'(\theta) > 0$ for $\theta \in (\xi, \pi/4)$. Therefore, f is decreasing on $(0, \xi)$ and f is increasing on $(\xi, \pi/4)$. In addition, $f(0) = f(\pi/4) = 0$. Therefore, $f(\theta) < 0$ for $(0, \pi/4)$.

4. Let $ABCD$ be a quadrilateral, with sides $AB = a$, $BC = b$, $CD = c$, where a, b, and c are fixed positive quantities. Prove that when the quadrilateral $ABCD$ has a maximum area, then $ABCD$ can be inscribed in a semicircle.

Solution.



Let θ denote $\langle ABC$ and ϕ denote $\langle BCD$. Since the lengths a, b, and c are given, the area $ABCD$ depends upon two independent variables θ and ϕ. A fixed value of ϕ will keep triangle BCD rigid, and the area $ABCD$ will then depend upon $\triangle ABD$.
Given any value of ϕ,

$$\text{the area of triangle } ABD = \frac{1}{2} AB \cdot BD \cdot \sin\langle ABD.$$

Since AB and BD are fixed, this area is maximum when $\langle ABD$ is a right angle.

Similarly, given a value of θ, the area $ABCD$ is maximized when $\langle ACD$ is a right angle. In other words, for the maximum area of $ABCD$, the points B and C must lie on a semicircle with AD as a diameter.

5. Define a sequence $\{x_n\}_{n=2}^{\infty}$ by

$$(n + x_n)[\sqrt[n]{2} - 1] = \ln 2.$$

Find $\lim_{n\to\infty} x_n$.

Solution.

Solving for x_n, we obtain

$$x_n = \frac{\ln 2}{\sqrt[n]{2} - 1} - n,$$

which is of the form $\infty - \infty$. Then, using the substitution $u = 1/n$ and l-Hospital's Theorem twice, we obtain

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{\ln 2 - n \cdot 2^{1/n} + n}{2^{1/n} - 1}$$

$$= \lim_{u\to 0} \frac{u \ln 2 - 2^u + 1}{u2^u - u}$$

$$= \lim_{u\to 0} \frac{1 - 2^u}{u2^u + \frac{2^u - 1}{\ln 2}}$$

$$= \lim_{u\to 0} \frac{-\ln 2}{u \ln 2 + 2}$$

$$= -\frac{1}{2} \ln 2.$$

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Session II

From *Crux Mathematicorum*, Problem 2532, April, 2000. Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea. Solution by Richard B. Eden, Ateneo de Manila University, Manila, The Philippines.

1. Suppose that a, b and c are positive real numbers satisfying $a^2+b^2+c^2=1$. Prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq 3+\frac{2(a^3+b^3+c^3)}{abc}.$$

Solution.

$$\begin{aligned}
&\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-3-\frac{2(a^3+b^3+c^3)}{abc}\\
&\qquad=\frac{a^2+b^2+c^2}{a^2}+\frac{a^2+b^2+c^2}{b^2}+\frac{a^2+b^2+c^2}{c^2}\\
&\qquad\quad-3-2\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)\\
&\qquad=a^2\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+b^2\left(\frac{1}{c^2}+\frac{1}{a^2}\right)+c^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\\
&\qquad\quad-2\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)\\
&\qquad=a^2\left(\frac{1}{b}-\frac{1}{c}\right)^2+b^2\left(\frac{1}{c}-\frac{1}{a}\right)^2+c^2\left(\frac{1}{a}-\frac{1}{b}\right)^2\geq 0,
\end{aligned}$$

with equality if and only if $a=b=c$.

2. Let $x_1>1$ be odd and define the sequence $\{x_n\}_{n=1}^{\infty}$ recursively by $x_n=x_{n-1}^2-2$, $n\geq 2$. Prove that for any pair of integers j, k satisfying $1\leq j<k$, the terms x_j, x_k are relatively prime.

Solution.

We have, immediately, $x_2 = x_1^2 - 2$, $x_3 = x_2^2 - 2 = (x_1^2 - 2)^2 - 2 = x_1^4 - 4x_1^2 + 2$, and in general, $x_k = P(x_j^2) \pm 2$, where $P(x_j^2)$ is a homogeneous polynomial in x_j^2, and $+2$ obtains if $k > j+1$ and -2 obtains if $k = j+1$. Let p be the largest prime that divides x_j, x_k. Then p must also divide 2, so either $p = 2$ or no prime p exists. But x_1 is odd, which implies all x_n's are odd. Hence, $(x_j, x_k) = 1$ is the only possibility.

3. Let $d(n)$ denote the number of divisors of n. Call n a round number if $m < n$ implies $d(m) < d(n)$. Prove that if n is round and

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$$

is the unique prime factorization of n with $p_1 < p_2 < \cdots < p_m$, then there is no prime missing between p_1 and p_m.

Solution.

First observe that the number of positive, integral divisors of a power of a prime p^α is just $\alpha + 1$ (the divisors are 1, p, p^2, ... , p^α). Suppose, now, that

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$$

is a round number, but that the jth natural prime P_j is missing from the chain of primes, $1 < j < r$. That is, $P_j < p_j$ and P_j is not a divisor of n. Form the new number n' by replacing $p_j^{\alpha_j}$ by $P_j^{\alpha_j}$.

Then, clearly, $n' < n$, but the number of positive, integral divisors of n' is, combinatorically,

$$(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_j + 1) \cdots (\alpha_r + 1).$$

This, however, is also the number of positive, integral divisors of n, so n could not have been a round number, a contradiction. Thus, no prime P_j is missing from the sequence.

Modification of a problem from the XI Italian Mathematical Olympiad 1995. The solution is a modification of a solution by Edward T. H. Wang.

4. In a class of 10 students no two of them have the same ordered pair (written and oral examinations) of scores in mathematics. We say that student A is better than B if his two scores are greater than or equal to the corresponding scores of B. The scores are integers between 1 and 5.

(a) Show that there exist three students A, B, and C such that A is better than B and B is better than C.

(b) Would the same be true for a class of 9 students?

Solution.

a) Let the ordered pairs be denoted by (a_i, b_i), $i = 1, 2, \ldots, 10$, where a_i and b_i denote, respectively, the scores of the ith student. For convenience of notation, write $(a_i, b_i) < (a_j, b_j)$ if the jth student is better than the ith student. Thus $(a_i, b_i) < (a_j, b_j)$ if and only if $a_i \le a_j$ and $b_i \le b_j$. Since there are 10 a_i's which are all in $\{1, 2, \ldots, 5\}$ we must have either

(1): For some $m \in \{1, 2, \ldots, 5\}$, $a_i = a_j = a_k = m$ where $i \ne j \ne k \ne i$,

or

(2): Every $m \in \{1, 2, \ldots, 5\}$ appears exactly twice as the first component in the ordered pairs.

Similarly, either

(3): For some $n \in \{1, 2, \ldots, 5\}$, $b_i = b_j = b_k = n$ where $i \ne j \ne k \ne i$,

or

(4): Every $n \in \{1, 2, \ldots, 5\}$ appears exactly twice as the second component in the ordered pairs.

In case (1), we would have three ordered pairs of the form (m, b_i), (m, b_j), and (m, b_k). Since b_i, b_j, and b_k must all be distinct, we may assume, without loss of generality, that $b_i < b_j < b_k$, and then $(m, b_i) < (m, b_j) < (m, b_k)$. Similarly, in case (3), we would have three ordered pairs of the form (a_i, n), (a_j, n), and (a_k, n). Assuming $a_I < a_j < a_k$, we then have $(a_i, n) < (a_j, n) < (a_k, n)$. If neither (1) nor (3) holds, then both (2) and (4) must hold and so every $l \in \{1, 2, \ldots, 5\}$ appears exactly twice as the first component and exactly twice as the second component in the ordered pairs. In particular, there must be ordered pairs of the form $(1, b_i)$ and $(1, b_j)$ for some $b_i < b_j$. Since 5 must appear twice as the second component and $b_i \ne b_j$, there must be at least one ordered pair of the form $(a_k, 5)$ for some $a_k \ne 1$. Then we have $(1, b_i) < (1, b_j) < (a_k, 5)$ and the conclusion follows.

(b) No. For example, if the 9 ordered pairs are $(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)$ and $(2, 5), (3, 4), (4, 3), (5, 2)$, then no two ordered pairs from the first five are comparable and no two ordered pairs from the other four are comparable. Hence the required "chain" of three students does not exist.

From Problem 2.2.11, *Berkeley Problems in Mathematics*, Springer, 1998, QA43.S695.

5. For a real 2×2 matrix

$$X = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

let $\|X\| = x^2 + y^2 + z^2 + t^2$, and define a distance function by $d(X, Y) = \|X - Y\|$. Let $\Sigma = \{X \mid \det(X) = 0\}$ and let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Find the minimum distance from A to Σ and exhibit a specific matrix $S \in \Sigma$ that achieves this minimum.

Solution.

The problem reduces to minimizing

$$F(x, y, z, t) = (x - 1)^2 + y^2 + z^2 + (t - 2)^2$$

subject to

$$g(x, y, z, t) = xt - yz = 0.$$

A Lagrange multiplier solution requires solving the five equations

$$\begin{aligned} 2(x-1) &= \lambda t \\ 2y &= -\lambda z \\ 2z &= -\lambda y \\ 2(t-2) &= \lambda x \\ xt &= yz. \end{aligned}$$

The second and third equations have only the trivial solution unless $\lambda^2 = 4$, and the first and fourth equations have no solution if $\lambda^2 = 4$, so we must have $y = z = 0$ and either $x = 0$ or $t = 0$. The two possible solutions are $(1, 0, 0, 0)$ and $(0, 0, 0, 2)$. Because $F(1, 0, 0, 0) = 4$ and $F(0, 0, 0, 2) = 1$, the minimum distance is 1 and occurs when

$$X = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$