2002 Missouri MAA Collegiate Mathematics Competition

Session I

1. Let $P \neq (0,0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at P will intersect the parabola at another point, say Q. Find the coordinates of P so that the average of the y-coordinates of P and Q is a minimum.

Solution.

Let $P = (p, p^2)$ and $Q = (q, q^2)$. The slope of the tangent line to the parabola at P is $2p$, so the slope of the normal line to the parabola at P is $-1/2p$. Thus, the equation of the normal line to the parabola at P is

$$y - p^2 = -\frac{1}{2p}(x - p).$$

Since $Q = (q, q^2)$ lies on the normal line,

$$q^2 - p^2 = -\frac{1}{2p}(q - p) \quad \text{or} \quad q + p = -\frac{1}{2p}.$$

Solving for q, we obtain

$$q = -p - \frac{1}{2p}.$$

Now the average of the y-coordinates of P and Q is

$$A = \frac{p^2 + q^2}{2} = \frac{p^2 + (-p - 1/2p)^2}{2} = p^2 + \frac{1}{2} + \frac{1}{8p^2}.$$

Differentiating this quantity with respect to p, we obtain

$$\frac{dA}{dp} = 2p - \frac{1}{4p^3} = \frac{1}{4p^3}(8p^4 - 1).$$

Setting this quantity equal to 0, we arrive at

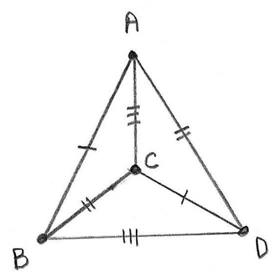
$$p = 2^{-3/4}.$$

Therefore, the coordinates of P so that the average of the y-coordinates of P and Q is a minimum are $(2^{-3/4}, 2^{-3/2})$.

From *Mathematical Gems II* by Ross Honsberger, Dolciani Mathematical Exposition, 1976, pp. 90–97.

2. A tetrahedron is called *isosceles* if the members of each pair of opposite edges are equal. This means, for tetrahedron $ABCD$, that $AB = CD$, $BC = AD$, and $AC = BD$.

(a) Prove that all four faces of an isosceles tetrahedron are congruent.
(b) Prove that if all four faces of a tetrahedron have the same perimeter, then the tetrahedron is isosceles.
(c) Prove that a tetrahedron is isosceles if and only if the sum of the face angles at each vertex is 180°.



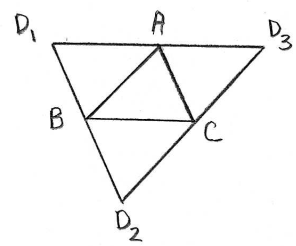
Solution.

(a) Side-Side-Side
(b) Denote opposite edges by a, a', b, b', and c, c'. Then from equal perimeters,

$$a + b + c = a + b' + c' = a' + b + c' = a' + b' + c$$

Eliminate a from the first two equations and a' from the last two. Solve each resulting equation for $b - b'$, giving $c - c' = c' - c$, which means $c = c'$. Similarly, $a = a'$ and $b = b'$. Thus the tetrahedron is isosceles.

(c) Imagine cutting $ABCD$ along the three edges that meet at D and flattening the tetrahedron out. This gives a hexagon $AD_1BD_2CD_3$.



Assume the sum of the face angles at each vertex is 180°. Then in the hexagon the angles at A, B, and C are each straight angles, making the hexagon really a triangle $D_1D_2D_3$ with A, B, and C as the midpoints of the sides. Thus, $AB = CD_2 = CD_3$, etc. and the tetrahedron is isosceles.

Assume the tetrahedron is isosceles. Then the faces are congruent, and so the face angles at a vertex are the same as the angles in a face triangle, which sum to 180°.

3. Let $\{x_n\}$ be the following sequence involving alternating square roots of 5 and 13:

$$x_1 = \sqrt{5}, \quad x_2 = \sqrt{5+\sqrt{13}}, \quad x_3 = \sqrt{5+\sqrt{13+\sqrt{5}}}, \quad x_4 = \sqrt{5+\sqrt{13+\sqrt{5+\sqrt{13}}}},$$

and so on. Prove that $\lim_{n\to\infty} x_n$ exists and determine its value.

Solution.

We see that $x_1, x_2 < 4$; assume also that $x_{2k-1}, x_{2k} < 4$. Then,

$$x_{2k+2} = \sqrt{5+\sqrt{13+x_{2k}}} < \sqrt{5+\sqrt{13+4}} < 4.$$

The argument is identical for x_{2k+1}. Hence, for each n, $x_n < 4$ by mathematical induction on n. In addition, the sequence increases monotonically. Therefore, by a standard limit theorem on sequences

$$\lim_{n\to\infty} x_n$$

must exist.

Let

$$L = \lim_{n\to\infty} x_n.$$

Then we have

$$L = \sqrt{5 + \sqrt{13 + L}}$$

or

$$L^4 - 10L^2 - L + 12 = 0.$$

One root is $L = 3$. Of the three remaining roots, one is positive (between 1 and 2) and the other two are complex. It follows that

$$\lim_{n\to\infty} x_n = 3.$$

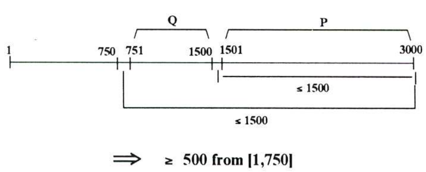
Convergence is fairly rapid; x_6 is already 2.999971.

From *From Erdős to Kiev* by Ross Honsberger, Dolciani Mathematical Exposition, 1995, pp. 177–179.

4. Does the set $X = \{1, 2, \ldots, 3000\}$ contain a subset A of 2000 integers in which no member of A is twice another member of A?

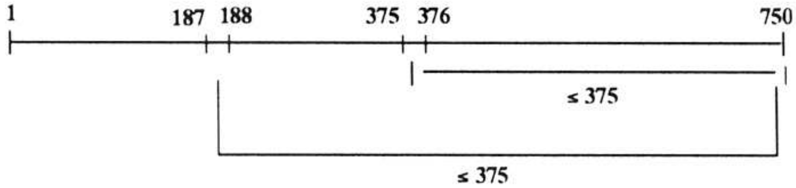
Solution.

Since twice any integer in the interval $P = [1501, 3000]$ is too big to belong to X, we could put these 1500 integers in A without worrying about doubling up on any of them. On the other hand, A certainly can't get more than 1500 integers from P since it only has 1500 altogether. Obviously, we have to be careful not to pick any integer in the interval $Q = [751, 1500]$ which is one-half an integer that is chosen from P. Clearly, each integer taken from Q negates the eligibility of its double in P, and it follows that, if k integers are taken from Q, then not more than $1500 - k$ can be selected from P, for a total of not more than 1500 altogether from the entire interval $Q \cup P = [751, 3000]$. Thus, in order to build up to 2000 integers in A, at least 500 must come from $[1, 750]$, the initial quarter of X.



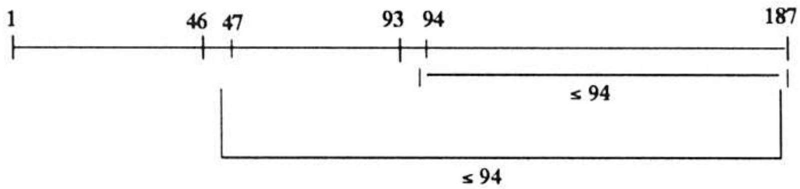
Repeated applications of this reasoning give the following results. In order to build up to 2000 integers in A, at least 125 must come from $[1, 187]$; at least 31 must come from $[1, 46]$; at least 8 from $[1, 11]$; and at least 2 from $[1, 2]$.

(i)



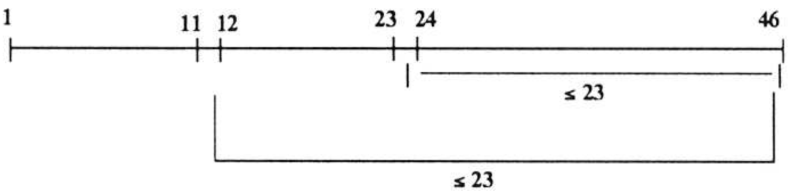
$\Longrightarrow$ $\geq$ 125 from [1,187]

(ii)



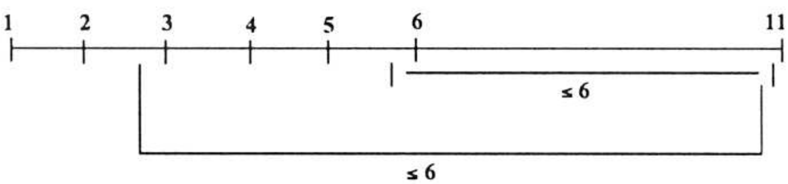
$\Longrightarrow$ $\geq$ 31 from [1,46]

(iii)



$\Longrightarrow$ $\geq$ 8 from [1,11]

(iv)



$\Longrightarrow$ $\geq$ 2 from [1,2],

which shows the undertaking is hopeless.

However, the proposal is not outrageous, for clearly A can be built up to

$$1500 + 375 + 94 + 23 + 6 + 1 = 1999 \quad \text{integers.}$$

This suggests that two-thirds the number of integers in X is a sharp cut-off point for the size of A, that is, that $|A|$ can be any number up to two-thirds the size of X, but not actually as big as $\frac{2}{3}|X|$. Applying our analysis to $X = \{1, 2, \ldots, 300\}$, however, reveals that A can have as many as 200 members:

$$A = \{\underbrace{1, 3, 4}_{3};\ \underbrace{10, 11, \ldots, 18}_{+9};\ \underbrace{38, 39, \ldots, 75}_{+38};\ \underbrace{151, 152, \ldots, 300}_{+150=200.}\}.$$

But, this set is as fully packed as possible, suggesting that the general result is rather $|A| \leq \frac{2}{3}|X|$.

From the above procedure, we can obtain the recursive formula

$$|A(N)| = \left\lceil \frac{N}{2} \right\rceil + \left| A\left(\left\lfloor \frac{N}{4} \right\rfloor \right) \right|,$$

where $|\cdot|$ denotes the number of elements in a set and $A(N)$ is a subset of $X = \{1, 2, \ldots, N\}$ with the no doubling property and having the largest number of elements.

Bruce Resnick (University of Illinois at Urbana-Champaign) has recently found the following pretty formula for the maximum size $f_r(n)$ of a subset of $\{1, 2, \ldots, n\}$ in which no element is r times another. Converting n to its representation in base r,

$$n = a_m a_{m-1} \cdots a_0,$$

then

$$f_r(n) = \frac{1}{r+1}\left(rn + \sum_{k=0}^{m} (-1)^k a_k \right).$$

5. Two right circular cylinders of radius r intersect at right angles to form a solid. This solid has four curved faces. Imagine one of these faces “rolled out flat”. Find equations of the boundary curves of this flattened face and also find its area.

Solution.

Let the x-axis be the axis of one cylinder and the y-axis be the axis of the other cylinder so that the center of the sold is the origin. Cross-sections perpendicular to the z-axis are squares with side $s = 2\sqrt{r^2 - z^2}$. Place the flattened face with its axis of symmetry on the horizontal axis, call it the w-axis, with the left end of the figure at the origin. This makes the range of w the interval $0 \leq w \leq \pi r$, and the distance from the w–axis to the top boundary curve is $\sqrt{r^2 - z^2}$. To relate z and w, note that if θ is the angle through which the solid has rolled, then $\cos\theta = z/r$ and $w = r\theta$, yielding

$$\sqrt{r^2 - z^2} = r \sin \frac{w}{r}.$$

The boundary curves, then are

$$f(w) = r \sin \frac{w}{r} \quad \text{and} \quad g(w) = -r \sin \frac{w}{r}.$$

The area is given by a simple integral:

$$\text{area} = \int_0^{\pi r} 2r \sin \frac{w}{r} dw = 4r^2.$$

This solid and its circumscribed cube have the same properties that Archimedes admired concerning the sphere and its circumscribed cylinder, namely, both the volumes and the surface areas are in the ratio 2 : 3.

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Session II

1. Seven golf balls, labeled 1 through 7, are correctly placed in corresponding boxes (one to a box), also labeled 1 through 7. The balls are now removed and then randomly returned to the boxes, one ball to a box. What is the probability that no ball will find its correct box?

Solution.

Let W_n denote the number of ways that n golf balls can be returned to n boxes with no ball finding its correct home. Suppose box 1 contains ball N ($N \neq 1$). There are then two cases.

CASE 1. Box N contains ball 1. Hence, the remaining $n-2$ balls are to be distributed among $n-2$ boxes, with no ball finding its correct home. There are W_{n-2} ways for this.

CASE 2. Box N doesn't contain ball 1. Then $n-1$ balls are to be similarly distributed among $n-1$ boxes; there are W_{n-1} ways here.

Since the choice of box 1 was arbitrary among the $n-1$ balls 2, 3, ..., n, the total number of ways is

$$W_n = (n-1)(W_{n-2} + W_{n-1}).$$

We have, clearly, $W_1 = 0$ and $W_2 = 1$. The recursion formula then gives, successively, $W_3 = 2$, $W_4 = 9$, $W_5 = 44$, $W_6 = 265$, and finally $W_7 = 1854$. But the number of permutations of 7 balls in 7 boxes is $7! = 5040$, so the desired probability is

$$P_7 = \frac{W_7}{7!} = \frac{1854}{5040} = \frac{103}{280} = 0.3679.$$

2.

(a) Prove that, for any positive integer n,

$$\sin n\theta = \binom{n}{1} \sin\theta \cos^{n-1}\theta - \binom{n}{3} \sin^3\theta \cos^{n-3}\theta + \binom{n}{5} \sin^5\theta \cos^{n-5}\theta - \cdots$$

and

$$\cos n\theta = \cos^n\theta - \binom{n}{2} \sin^2\theta \cos^{n-2}\theta + \binom{n}{4} \sin^4\theta \cos^{n-4}\theta - \cdots .$$

(b) Prove that, for all x in the interval $[-1, 1]$ and any positive integer n, the function

$$T_n(x) = \cos(n \cos^{-1} x)$$

is a polynomial in x of degree n and leading coefficient 2^{n-1}.

Solution.

For part (a), from DeMoivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta,$$

we expand the left side using the binomial theorem and equate real and imaginary parts.

For part (b), use part (a) with $\theta = \cos^{-1} x$, $\cos\theta = x$, and $\sin\theta = \sqrt{1-x^2}$ to get

$$\cos(n\cos^{-1} x) = x^n - \binom{n}{2}(1-x^2)x^{n-2} + \binom{n}{4}(1-x^2)^2 x^{n-4} - \cdots .$$

It is clear that $T_n(x)$ is a polynomial of degree n, and that the leading coefficient is

$$1 + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{2k}$$

where

$$2k = \begin{cases} n, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd.} \end{cases}$$

Now, since

$$2^n = (1+1)^n = 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

and

$$0 = (1-1)^n = 1 - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n}$$

we can add and get the leading coefficient above equal to 2^{n-1}.

From *Mathematical Gems II* by Ross Honsberger, Dolciani Mathematical Exposition, 1976, pp. 18–20.

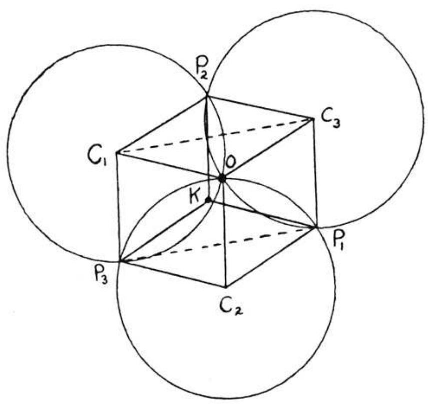
3. Suppose three equal circles, each of radius r, pass through a common point O and have three other pairwise intersections at P_1, P_2, and P_3. Prove that the circle containing P_1, P_2, and P_3 also has radius r.

Solution.

Label the three circles 1, 2, 3, with centers C_1, C_2, C_3, respectively. Denote the point of intersection common only to circles i and j as P_k, $i, j, k = 1, 2, 3$, $i \neq j \neq k$. The circle through points C_1, C_2, C_3 has center O and radius r since the distance from O to each C_i is r.

$C_1OC_2P_3$, $C_1OC_3P_2$, and $C_2OC_3P_1$ are rhombi whose all four sides are equal to r. Thus, C_1P_3 and OC_2 are equal and parallel, as are OC_2 and C_3P_1. This implies C_1P_3 and C_3P_1 are equal and parallel, so that $C_1P_3P_1C_3$ is a parallelogram, and thus, C_1C_3 and P_1P_3 are equal and parallel. Similarly, C_1C_2 and P_1P_2, as well as C_2C_3 and P_2P_3, are equal and parallel.

Therefore, $\triangle C_1C_2C_3$ and $\triangle P_1P_2P_3$ are congruent which implies the circumcircles of each are equal, and each circle has radius r.



4. $ABCDE$ is a regular pentagon of side s, and P is any point in the interior of $ABCDE$. Line segments are drawn from P perpendicular to each of the five sides. Denote the sum of the lengths of these five perpendiculars by S. Prove that S is independent of the location of P, and find S in terms of s.

Solution.

Let P be an arbitrary point in the interior. Then

$$\text{area } \triangle PAB = \frac{1}{2} \cdot s \cdot h_1,$$

where h_1 is the length of the perpendicular from P to AB. Similarly,

$$\text{area } \triangle PBC = \frac{1}{2} \cdot s \cdot h_2,$$

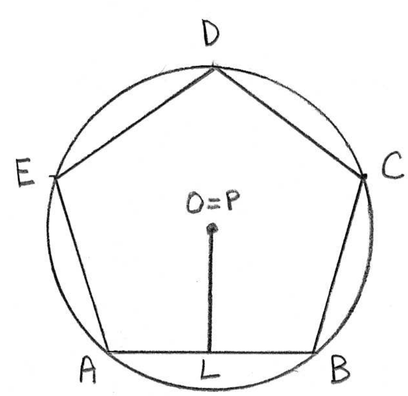
where h_2 is the length of the perpendicular from P to BC. Continue this process for $\triangle PCD$, $\triangle PDE$ and $\triangle PEA$. The sum of the areas of these triangles is the area of the pentagon, that is,

$$\frac{1}{2}s(h_1 + h_2 + h_3 + h_4 + h_5) = \text{area of pentagon}.$$

Therefore,

$$S = h_1 + h_2 + h_3 + h_4 + h_5$$

is independent of the position of P because P was arbitrary.



To compute S, let $P = O =$ center of the pentagon. Let L be the point on side AB so that OL is perpendicular to side AB. Then $S = 0.5L$ and

$$\angle AOB = 72^\circ, \qquad \angle AOL = 36^\circ, \qquad \frac{AL}{OL} = \tan 36^\circ,$$

$$AL = OL \tan 36^\circ, \qquad OL = AL \cot 36^\circ = \frac{s}{2} \cot 36^\circ.$$

Therefore,

$$S = 5OL = s\left(\frac{5}{2}\cot 36^\circ\right).$$

From the 1980 All-Union Russian Olympiad.

5. Let $p(n)$ denote the product of the (decimal) digits of the positive integer n. Consider the sequences, beginning at any arbitrary positive integer, in which succeeding terms are obtained by adding to the previous term the product of its digits:

$$n_0 = n, \quad \text{and for} \quad r \geq 0, \quad n_{r+1} = n_r + p(n_r).$$

Is there an initial integer n for which the sequence continues to increase indefinitely?

Solution.

The answer is no.

Note that a digit equal to zero anywhere in n_r results in $p(n_r) = 0$, leading to $n_{r+1} = n_r$ and to n_r repeating indefinitely; otherwise, $p(n_r)$ is positive and the sequence increases.

The list of consecutive numbers containing a 0 at the beginning of the set of integers that contain a given number of digits is

$$\{100, 101, \ldots, 109\}$$
$$\{1000, 1001, \ldots, 1099\}.$$

In general, for $k \geq 3$, the k-digit integers begin with 10^{k-2} consecutive numbers containing a 0 (actually there are more than this but we do not need greater accuracy). Now if $p(n_r)$ were to increase n_r to any value in this initial segment of k-digit numbers containing a 0, the sequence would never get any bigger.

For a $(k-1)$-digit number n_r, $p(n_r)$ cannot exceed 9^{k-1}. If it should ever happen that the initial segment of 10^{k-2} k-digit numbers containing a 0 were too great for $p(n_r)$ to get over, that is, if ever $10^{k-2} > 9^{k-1}$, then all sequences with terms containing $k-1$ or fewer digits could never muster the increment necessary to reach any k-digit integer that did not contain a zero – those that managed to survive to the k-digit level would remain in one of our 10^{k-2} consecutive k-digit integers containing a 0. Starting with

$$10^{k-2} > 9^{k-1}$$

and taking logs, we see that

$$k - 2 > (k-1)\log 9$$

$$k > \frac{2 - \log 9}{1 - \log 9} \doteq 22.85.$$

Thus, no sequence that starts below 23-digit numbers can survive beyond our 10^{21} consecutive 23-digit integers containing a 0, and sequences that start at $m \geq 23$-digit numbers cannot get past the first 10^{m-1} integers with $m+1$ digits.