2001 Missouri MAA Collegiate Mathematics Competition

Session I

1. Let $P \neq (0,0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at P will intersect the parabola at another point, say Q. Find the coordinates of P so that the area bounded by the normal line and the parabola is a minimum.

Solution.

Let $P = (p, p^2)$ and $Q = (q, q^2)$. Without loss of generality, let $p > 0$. The slope of the tangent line to the parabola at P is $2p$ so the slope of the normal line to the parabola at P is $-1/2p$. Thus, the equation of the normal line to the parabola at P is

$$y - p^2 = -\frac{1}{2p}(x - p).$$

Since $Q = (q, q^2)$ lies on the normal line,

$$q^2 - p^2 = -\frac{1}{2p}(q - p) \quad \text{or} \quad q + p = -\frac{1}{2p}.$$

Solving for q we obtain

$$q = -p - \frac{1}{2p}.$$

Now the area bounded by the normal line and the parabola (as a function of p) is

$$A = \int_q^p \left(-\frac{1}{2p}(x - p) + p^2 - x^2 \right) dx.$$

Integrating, substituting the p expression for q, and simplifying, we obtain

$$A = -\frac{1}{4p}(x-p)^2 + p^2 x - \frac{1}{3}x^3 \Big|_q^p = \frac{2}{3}p^3 + \frac{1}{4p}(q-p)^2 - p^2 q + \frac{1}{3}q^3$$

$$= \frac{2}{3}p^3 + \frac{1}{4p}\left(-2p - \frac{1}{2p}\right)^2 - p^2\left(-p - \frac{1}{2p}\right) + \frac{1}{3}\left(-p - \frac{1}{2p}\right)^3$$

$$= \frac{4}{3}p^3 + p + \frac{1}{4p} + \frac{1}{48p^3}.$$

Differentiating with respect to p yields

$$\frac{dA}{dp} = 4p^2 + 1 - \frac{1}{4p^2} - \frac{1}{16p^4}$$

$$= \frac{1}{16p^4}(64p^6 + 16p^4 - 4p^2 - 1)$$

$$= \frac{1}{16p^4}(4p^2 - 1)(16p^4 + 8p^2 + 1).$$

Setting the derivative equal to 0 results in $p = 1/2$ so the coordinates of P so that the area bounded by the normal line and the parabola is a minimum are $(1/2, 1/4)$.

2. Let $\{x_i\}$ denote any finite sequence with the following properties:

(a) $x_i \in \{-2, 1, 2\}$ for each x_i,
(b) $\sum_i x_i = 29$,
(c) $\sum_i x_i^2 = 59$.

In considering the family of all such sequences, let $M = \max\{\sum_i x_i^3\}$ and $m = \min\{\sum_i x_i^3\}$. Determine M/m.

Solution.

In a sequence $\{x_i\}$, let a = number of -2's, b = number of 1's, and c = number of 2's. Properties (b) and (c) then yield

$$\begin{cases} -2a + b + 2c = 29 \\ 4a + b + 4c = 59, \end{cases}$$

or upon elimination of b: $3a + c = 15$. To obtain M the sum $\sum_i x_i^3$ must be dominated by the 2's. The maximum value of c consistent with $3a + c = 15$ is $c = 15$. But this leads to $b = -1$; hence, $c = 12$, $a = 1$, $b = 7$, and $M = 1(-2)^3 + 7(1)^3 + 12(2)^3 = 95$.

To obtain m the sum $\sum_i x^3$ must be dominated by the -2's. The maximum value of a consistent with $3a + c = 15$ is $a = 5$, so $b = 39$ and $c = 0$. Hence, $m = 5(-2)^3 + 39(1)^3 + 0(2)^3 = -1$.

It follows that $M/m = -95$.

3. Let a, b, and c be the sides of a triangle with perimeter 2. Prove that

$$3/2 < a^2 + b^2 + c^2 + 2abc < 2.$$

Solution.

Since the perimeter is 2, no side can be greater than 1, so the area,

$$A = \frac{1}{2} bc \sin \alpha,$$

is less than 1/2. Then $A^2 = (1-a)(1-b)(1-c)$ implies

$$\begin{gathered}
0 < (1-a)(1-b)(1-c) < 1/4, \\
0 < 1 - (a+b+c) + (ab+ac+bc) - abc < 1/4, \\
0 < 1 - 2 + (ab+ac+bc) - abc < 1/4, \\
1 < (ab+ac+bc) - abc < 5/4,
\end{gathered}$$

and

$$2 < 2(ab+ac+bc) - 2abc < 5/2.$$

Finally,

$$a^2 + b^2 + c^2 + 2abc = (a+b+c)^2 + (2abc - 2(ab+ac+bc)),$$

so,

$$2^2 - 5/2 < a^2 + b^2 + c^2 + 2abc < 2^2 - 2,$$

or,

$$3/2 < a^2 + b^2 + c^2 + 2abc < 2.$$

4. Find the sum

$$S = \sum_{k=1}^{\infty} \frac{k^2}{3^k}.$$

Solution.

Define

$$F(x) = \sum_{k=1}^{\infty} \frac{x^{k+1}}{3^k}, \quad |x| < 3.$$

As this series is geometric, we have, explicitly,

$$F(x) = \frac{x^2}{3-x}.$$

Differentiation gives

$$F'(x) = \sum_{k=1}^{\infty} \frac{(k+1)x^k}{3^k} = \frac{6x - x^2}{(3-x)^2},$$

so

$$\sum_{k=1}^{\infty} \frac{kx^k}{3^k} = \frac{6x - x^2}{(3-x)^2} - \frac{F(x)}{x} = \frac{3x}{(3-x)^2}.$$

A second differentiation gives, from the previous line

$$\sum_{k=1}^{\infty} \frac{k^2 x^{k-1}}{3^k} = \frac{3(x+3)}{(3-x)^3},$$

so

$$S = \sum_{k=1}^{\infty} \frac{k^2}{3^k} = \frac{3(1+3)}{(3-1)^3} = \frac{3}{2}.$$

5. A set of five cubical dice has the following properties:

(a) On each face of each die is a 3-digit integer. No two integers on a given face are the same.
(b) Every integer has a nonzero hundred's digit.
(c) The sum, when the dice are rolled, of the five integers is a 4-digit integer.
(d) Whenever the dice are rolled, their sum S can be found quickly as follows: the sum of the unit's digits of the five dice is the last two digits of S, and the first two digits of S are 50 minus the sum of the unit's digits.

For example, if the dice come up 189, 256, 275, 845, and 168, the sum of the unit's digits is $9 + 6 + 5 + 5 + 8 = 33$, so the value of S is 1733, since $50 - 33 = 17$.

Explain, justifying your statements, how such a set of dice can be constructed.

Solution.

Since the last two digits of S equals the sum of the unit's digits of the five dice, the sum of the ten's digits must be a multiple of 10. This can be accomplished in a number of ways, such as making the ten's digit on all faces of each die the same: die 1 ten's digit = 4, die 2 ten's digit = 5, die 3 ten's digit = 6, die 4 ten's digit = 7, die 5 ten's digit = 8 will make the sum of the ten's digits always be 30. The quick-sum property, assuming the sum of the ten's digits is 30, gives

$$S = 100\left(\sum_{i=1}^{5} h_i\right) + 300 + \sum_{i=1}^{5} u_i = 100\left(50 - \sum_{i=1}^{5} u_i\right) + \sum_{i=1}^{5} u_i.$$

Here, h_i and u_i are the hundred's and unit's digits on die i. This equation reduces to

$$\sum_{i=1}^{5}(h_i + u_i) = 47.$$

So, whenever the dice are rolled, the sum of the 5 hundred's digits plus the sum of the 5 unit's digits must be 47. This can occur if the hundred's digit plus the unit's digit on all faces of each die is a constant and the five constants add to 47. One possibility for the five constants is 13, 8, 9, 7, and 10, which was the case in the example.

This problem can be solved in more generality - there is nothing sacred about the 50, for instance. A pertinent result is that the above solution guarantees that the difference between any two faces of a given die is a multiple of 99, and this property can be used as a starting point to solve the problem.

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Session II

1. Circle B lies wholly in the interior of circle A. Find the loci of points equidistant from the two circles?

Solution.

The locus of points is an ellipse. Let circle A have radius r_A and center (x_A, y_A) and circle B have radius r_B and center (x_B, y_B). If (x, y) is a point equidistant from the two circles, then the distance (x, y) is from outer circle A,

$$r_A - \sqrt{(x - x_A)^2 + (y - y_A)^2}$$

is equal to the distance (x, y) is from inner circle B,

$$\sqrt{(x - x_B)^2 + (y - y_B)^2} - r_B.$$

That is,

$$r_A - \sqrt{(x - x_A)^2 + (y - y_A)^2} = \sqrt{(x - x_B)^2 + (y - y_B)^2} - r_B.$$

Therefore,

$$r_A + r_B = \sqrt{(x - x_A)^2 + (y - y_A)^2} + \sqrt{(x - x_B)^2 + (y - y_B)^2}.$$

But this is exactly the equation of an ellipse.

2. Show that if x, y, and z are positive reals such that $x + y + z = 1$, then

$$\left(\frac{1}{x} - 1\right)\left(\frac{1}{y} - 1\right)\left(\frac{1}{z} - 1\right) \geq 8.$$

Solution.

$$\left(\frac{1}{x} - 1\right)\left(\frac{1}{y} - 1\right)\left(\frac{1}{z} - 1\right) = \frac{1-x}{x} \cdot \frac{1-y}{y} \cdot \frac{1-z}{z} = \frac{(y+z)(z+x)(x+y)}{xyz}.$$

The problem then reduces to showing that

$$(x+y)(y+z)(z+x) \geq 8xyz.$$

But, by the A.M.–G.M. inequality,

$$\begin{aligned} x+y &\geq 2\sqrt{xy} \\ y+z &\geq 2\sqrt{yz} \\ z+x &\geq 2\sqrt{zx}. \end{aligned}$$

Multiplying the left and right sides of the inequalities, we obtain

$$(x+y)(y+z)(z+x) \geq 8xyz$$

which establishes the result.

3. A convex decagon and all of its diagonals are drawn. How many *interior* points of intersection of the diagonals are there, if it is assumed that no 3 diagonals share a common *interior* point?

Solution.

Let $N(n)$ be the number of such interior intersection points in a convex n-gon. Label the vertices 1, 2, 3, ..., n. Any four labels selected from this list (e.g., $(1, 3, n, 6)$) define a quadrilateral, which is convex because the n-gon is convex. The quadruple of integers therefore also defines uniquely an interior intersection point of the diagonals of the quadrilateral, and hence also of the n-gon.

The total number of interior intersection points of diagonals of the n-gon (assuming no 3 of the diagonals meet at a common point) is thus the number of ways that four integers can be selected (without regard to order) from a list of n distinct integers. Therefore,

$$N(n) = \binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24},$$

which for the decagon gives $N(10) = 210$.

From *From Erdős to Kiev* by Ross Honsberger, Dolciani Mathematical Exposition, 1995, pp. 8–9.

4. No matter what n real numbers $x_1, x_2, \ldots, x_n$ may be selected in the closed unit interval $[0, 1]$, prove that there always exists a real number x in this interval such that the average unsigned distance from x to the x_i's is exactly $1/2$.

Solution.

Adopting a perfectly straightforward approach, look at the function

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} |x - x_i|.$$

At $x = 0$, we have (since $x_i \geq 0$)

$$f(0) = \frac{1}{n}\sum_{i=1}^{n} |-x_i| = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

For $x = 1$, we get (since $x_i \in [0, 1]$)

$$\begin{aligned} f(1) &= \frac{1}{n}\sum_{i=1}^{n} |1 - x_i| \\ &= \frac{1}{n}\sum_{i=1}^{n}(1 - x_i) \\ &= \frac{1}{n}\left(n - \sum_{i=1}^{n} x_i\right) \\ &= 1 - f(0), \end{aligned}$$

and we have the crucial relation

$$f(0) + f(1) = 1.$$

In view of this, the two values $f(0)$ and $f(1)$ are either each equal to $1/2$, providing two solutions to the problem, or their values *straddle* $1/2$, in which case the continuity of the function implies $f(x) = 1/2$ for some x between 0 and 1.

From *From Erdős to Kiev* by Ross Honsberger, Dolciani Mathematical Exposition, 1995, pp. 6–8.

5. Consider the polynomial

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + 1,$$

where $a_i \geq 0$. If the equation $f(x) = 0$ happens to have n real roots, is it not remarkable that the value of $f(2)$ must then be at least 3^n? Prove this unlikely consequence: $f(2) \geq 3^n$.

Solution.

Since all the coefficients a_i are ≥ 0, the substitution of any nonnegative number for x would make $f(x)$ at least 1, implying that all the roots of $f(x) = 0$ must be negative numbers, say $-r_1, -r_2, \ldots, -r_n$. Using these roots to factor $f(x)$, we obtain

$$\begin{aligned} f(x) &= x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + 1 \\ &= (x + r_1)(x + r_2)(x + r_3) \cdots (x + r_n) \\ &= x^n + (r_1 + r_2 + \cdots + r_n) x^{n-1} + (r_1 r_2 + r_1 r_3 + \cdots) x^{n-2} + \cdots + r_1 r_2 \cdots r_n. \end{aligned}$$

Thus each coefficient a_k is given by

$$\begin{aligned} a_k = &\text{ the sum of the } \binom{n}{k} \text{ products of the roots taken } k \text{ at a time} \\ = &\sum r_{i_1} r_{i_2} \cdots r_{i_k}. \end{aligned}$$

Also, the absolute term $r_1 r_2 \cdots r_n = 1$.

Now, applying the arithmetic mean - geometric mean inequality to the $\binom{n}{k}$ terms that make up a_k, we obtain

$$\frac{a_k}{\binom{n}{k}} = \frac{\sum r_{i_1} r_{i_2} \cdots r_{i_k}}{\binom{n}{k}} \geq \left(\prod r_{i_1} r_{i_2} \cdots r_{i_k} \right)^{1/\binom{n}{k}},$$

from which we get

$$a_k \geq \binom{n}{k}\left(\prod r_{i_1} r_{i_2} \cdots r_{i_k}\right)^{1/\binom{n}{k}}.$$

Admittedly, this doesn't seem to be very promising. However, this awkward product melts away completely with the brilliant observation that, since there is no preference for one r_i over another, it follows that each r_i occurs in the product the same number of times altogether, with the result that, for some positive integer t,

$$\prod r_{i_1} r_{i_2} \cdots r_{i_k} = (r_1 r_2 \cdots r_n)^t = 1^t = 1.$$

Therefore,

$$a_k \geq \binom{n}{k}.$$

Now, k only runs from 1 to $n-1$, and so if we set $a_0 = a_n = 1$, then a_0 and a_n would respectively equal $\binom{n}{0}$ and $\binom{n}{n}$, making

$$a_k \geq \binom{n}{k}$$

for all $k = 0, 1, 2, \ldots, n$. Thus we can write

$$f(x) = \sum_{k=0}^{n} a_k x^{n-k},$$

and the value of $f(2)$ is given by

$$\sum_{k=0}^{n} a_k 2^{n-k}.$$

Since $a_k \geq \binom{n}{k}$, then (by the binomial theorem)

$$f(2) \geq \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 2^{n-k} = (1+2)^n = 3^n.$$