2000 Missouri MAA Collegiate Mathematics Competition

Session I

1. Let $P \neq (0,0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at P will intersect the parabola at another point, say Q. Find the coordinates of P so that the distance between the x-coordinates of P and Q is a minimum.

Solution.

Let $P = (p, p^2)$ and $Q = (q, q^2)$. The slope of the tangent line to the parabola at P is $2p$ so the slope of the normal line to the parabola at P is $-1/2p$. Thus, the equation of the normal line to the parabola at P is

$$y - p^2 = -\frac{1}{2p}(x - p).$$

Since $Q = (q, q^2)$ lies on the normal line,

$$q^2 - p^2 = -\frac{1}{2p}(q - p) \quad \text{or} \quad q + p = -\frac{1}{2p}.$$

Now the distance between the x-coordinates of P and Q is

$$D = \left| 2p + \frac{1}{2p} \right|.$$

Assuming $p > 0$,

$$D = 2p + \frac{1}{2p}.$$

Using the AM–GM Inequality, it is known that for all $x > 0$,

$$x + \frac{1}{x} \geq 2,$$

with equality when $x = 1$. Therefore, $p = 1/2$ if $p > 0$ or $p = -1/2$. Hence, the coordinates of P resulting in a minimum distance between the x-coordinates of P and Q are

$$\left(\pm\frac{1}{2}, \frac{1}{4}\right).$$

From *Crux Mathematicorum*, Problem 323, Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York and M. S. Klamkin, University of Alberta, 4 (1978), 255–256.

2. If $xyz = (1-x)(1-y)(1-z)$ where $0 \leq x, y, z \leq 1$, show that

$$x(1-z) + y(1-x) + z(1-y) \geq \frac{3}{4}.$$

Solution by Basil C. Rennie, James Cook University of North Queensland, Australia.

From $0 \leq x \leq 1$ and $(x - 1/2)^2 \geq 0$, we get $0 \leq x(1-x) \leq 1/4$. From this and two similar results,

$$xyz(1-x)(1-y)(1-z) \leq \frac{1}{64}. \tag{1}$$

By the given relation, the left side of (1) is $(xyz)^2$; so $xyz \leq 1/8$. Using the relation again yields

$$x(1-z) + y(1-x) + z(1-y) = 1 - 2xyz \geq \frac{3}{4}.$$

Equality occurs when $x = y = z = 1/2$.

3. Let $n \geq 3$ points be given in the plane. Prove that three of them form an angle which is at most π/n.

Solution.

Suppose the n points are labeled as P_j, $1 \leq j \leq n$. Note that if any 3 of them are collinear then an angle of measure zero exists, so we may assume no 3 of them are collinear. Let P be the convex polygon obtained as the convex hull of the n points. Then each of the points is either a vertex or an interior point of P and at least 3 of them are vertices. Assume then that P_1, P_2, P_n are adjacent vertices of P with P_1 "between" P_2 and P_n. Then (by relabeling points as needed) we have

$$\begin{aligned}\pi &= m(\angle P_1P_2P_n) + m(\angle P_1P_nP_2) + m(\angle P_2P_1P_n) \\ &= m(\angle P_1P_2P_n) + m(\angle P_1P_nP_2) + \sum_{i=2}^{n-1} m(\angle P_iP_1P_{i+1}).\end{aligned}$$

So by the pigeonhole principle, at least one of these n angles has measure less than or equal to π/n.

From Joh. Bernoulli, 1697.

4. Justify as far as you can, the equality

$$\int_0^1 x^x\,dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \cdots.$$

Solution.

Formally, we have (ln denotes the natural logarithm)

$$\int_0^1 x^x\,dx = \int_0^1 e^{x\ln x}\,dx = \sum_{n=0}^{\infty} \frac{1}{n!}\int_0^1 (x\ln x)^n\,dx.$$

If $m \le n$, we have, using integration by parts,

$$\int_0^1 x^n(\ln x)^m\,dx = -\frac{m}{n+1}\int_0^1 x^n(\ln x)^{m-1}\,dx,$$

and so

$$\int_0^1 (x\ln x)^n\,dx = (-1)^n\frac{n!}{(n+1)^n}\int_0^1 x^n\,dx = \frac{(-1)^n n!}{(n+1)^{n+1}}.$$

We get, then,

$$\int_0^1 x^x \, dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n} = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \cdots .$$

Note. Bernoulli's 1697 solution ended here. The need to deal with uniform convergence-type questions was not recognized until later.

Let $\{f_n(x)\}_{n=0}^{\infty}$ be the sequence of functions defined by

$$f_n(x) = \begin{cases} (x \ln x)^n / n!, & n \geq 1, \ 0 < x \leq 1 \\ 1, & n = 0, \ 0 \leq x \leq 1. \end{cases}$$

n-fold application of L'Hôpital's theorem gives

$$\lim_{x \to 0+} f_n(x) = 0, \quad (n > 0),$$

so $f_n(x)$ is continuous, and therefore integrable, on $[0,1]$. On the interval $(0,1)$ $x \ln x$ is negative and has a single local minimum of $-e^{-1}$. It follows that on $[0,1]$,

$$|f_n(x)| \leq \frac{1}{e^n n!}$$

for $n \geq 0$. One sees easily that

$$|f_n(x)| < \frac{1}{n^2}$$

for all $n \geq 1$. But

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, so by the Weierstrass M-Test the sequence $\{f_n(x)\}_{n=0}^{\infty}$ converges uniformly on $[0,1]$. This, in turn, guarantees that

$$\sum_{n=0}^{\infty}\int_0^1 f_n(x)dx = \int_0^1\left(\sum_{n=0}^{\infty} f_n(x)\right)dx = \int_0^1 x^x\,dx.$$

5. Show that a polynomial in x with real coefficients which takes rational values for rational x and (real) irrational values for (real) irrational x must be linear.

Solution.

Note that the polynomial cannot be constant. Denote the polynomial by $f(x)$ and its degree by n. Since $x \in \mathbb{Q}$ implies $f(x) \in \mathbb{Q}$, we know that $f(0) = y_0$, $f(1) = y_1$, $\ldots$, $f(n) = y_n$ are all rational. Now consider

$$g(x) = \sum_{i=0}^{n} y_i L_i(x),$$

where

$$L_i(x) = \prod_{\substack{k=0\\k\neq i}}^{n} \frac{(x-k)}{(i-k)}.$$

Since $f(x)$ and $g(x)$ are both polynomials of degree n which agree at the $n+1$ values $0, 1, \ldots, n$, $f(x)$ and $g(x)$ must have the same coefficients, so $f(x)$ must have rational coefficients.

Now assume to the contrary that $n > 1$. Since the coefficients of $f(x)$ are rational, we can write

$$f(x) = (a_n x^n + \cdots + a_1 x + a_0)/d,$$

where a_i and d are integers. We will derive a contradiction by finding a rational value r such that $f(x) = r$ has a real irrational solution. Let $r = (p + a_0)/d$, with p a prime number to be determined. Without loss of generality, we may assume that $a_n > 0$. For sufficiently large p, $f(x) = r$ has a positive root which is approximately

$$\left(\frac{p}{a_n}\right)^{1/n}.$$

But by the Rational Root Test, the only positive rational roots of $f(x) = r$ are $1/\lambda$ or p/λ, where λ is a divisor of a_n. But for sufficiently large p, the former are too small while the latter are too large, so $f(x) = r$ has a real irrational root. Contradiction.

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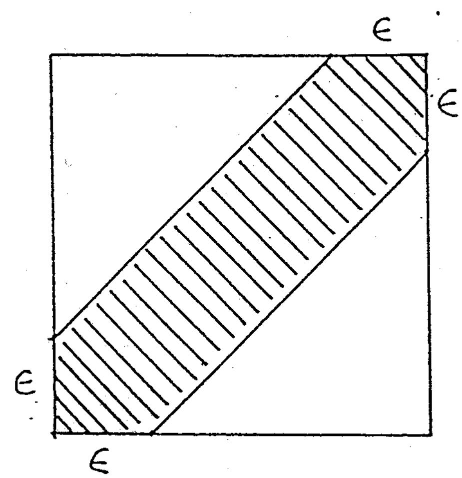
Session II

1. Two points are chosen at random (with a uniform distribution) from the unit interval $[0, 1]$. What is the probability that the points will be within a distance of ϵ of each other?

Solution.

Clearly if $\epsilon \geq 1$, then the probability is 1. If $0 \leq \epsilon < 1$, then the probability is the area of the region defined by $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $|x - y| \leq \epsilon$ (divided by the area of the unit square). But this is $1 - 2 \cdot (1 - \epsilon)^2/2 = 2\epsilon - \epsilon^2$.



From Macalester College Problem of the Week # 820.

2. Write Pascal's Triangle as an infinite array as follows:

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 3 & 6 & 10 & 15 & \cdots \\ 1 & 4 & 10 & 20 & 35 & \cdots \\ 1 & 5 & 15 & 35 & 70 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

where the first row and first column consist entirely of ones and every other entry is formed by taking the sum of the entry to the left and the entry above. For each positive integer

n, let D_n denote the n by n matrix formed by the first n rows and first n columns of the array. What is the determinant of D_n? Prove your answer.

Solution.

Direct calculations for small values of n suggest that $\det(D_n) = 1$ for all n.

Outline of Proof:

Perform the following elementary row operations on D_n in the order specified to change rows n through 2:

1. Subtract the $(n-1)$st row from the nth row,
2. Subtract the $(n-2)$nd row from the $(n-1)$st row,

$$\vdots$$

$(n-1)$. Subtract the 1st row from the 2nd row.

Because of the way the array is constructed, this will translate each column one position to the right, and will make the first column a one followed by $(n-1)$ zeros. Repeating this process for rows n through 3, then for rows n through 4, etc. gives an upper triangular matrix with ones on the diagonal, which clearly has determinant 1.

Alternatively, expansion by minors of the first column after the first set of row operations yields $\det(D_n) = \det(D_{n-1})$. This and the values for small n give the result.

3. Given an $n \times n$ checkerboard with the four corners removed, characterize for which n this deleted board can be covered with 3×1 rectangles.

Solution.

There are three cases to consider. If $n \equiv 0 \pmod 3$, then there are $n^2 - 4$ squares in the deleted board but since this is not a multiple of 3, such a covering is impossible. If $n \equiv 2 \pmod 3$, the board can be decomposed into a large central $n \times (n-2)$ rectangle and two narrow peripheral $(n-2) \times 1$ rectangles each of which can clearly be covered by 3×1 rectangles. If $n \equiv 1 \pmod 3$ i.e. $n = 3k+1$, label the squares 1, 2, or 3 as shown in the figure. Any 3×1 rectangle put on the board will cover exactly one of each of these numbers, so if a covering is possible, there must be the same number of 1's, 2's, and 3's. However, there are $4 + 7 + \cdots + (n-3) + (n-2) + (n-3) + \cdots + 7 + 4 = 3k^2 + 2k - 3$ squares labeled with a 1. If there were an equal number of squares with each label there would be $[(3k+1)^2 - 4]/3 = 3k^2 + 2k - 1$ squares labeled with a 1. Contradiction.

	2	3	1	2	3	
2	3	1	2	3	1	2
3	1	2	3	1	2	3
1	2	3	1	2	3	1
2	3	1	2	3	1	2
3	1	2	3	1	2	3
	2	3	1	2	3	

4. For $n \geq 2$, let $x_1, \ldots, x_n$ be non-zero real numbers whose sum is zero. Show that there are i, j with $1 \leq i < j \leq n$ such that

$$1/2 \leq |x_i/x_j| \leq 2.$$

Solution.

Let $a = \min\{x_i\}$ and $b = \max\{x_i\}$. We may assume (by replacing each x_i with $-x_i$ if necessary) that $|a| > b$. If $|a| \leq 2b$, then $1 \leq |a/b| \leq 2$ and we are done. So assume that $b < |a|/2$.

Now consider the (infinitely many) intervals

$$\ldots, \left[\frac{|a|}{2^n}, \frac{|a|}{2^{n-1}}\right), \ldots, \left[\frac{|a|}{8}, \frac{|a|}{4}\right), \left[\frac{|a|}{4}, \frac{|a|}{2}\right).$$

One of these intervals must contain 2 of the positive x_i values for otherwise

$$\sum_{x_i>0} x_i < \frac{|a|}{2} + \frac{|a|}{4} + \cdots = |a| \cdot \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) = |a|,$$

which is impossible since

$$\sum_{i=1}^{n} x_i = 0.$$

So there exist $i \neq j$ such that x_i and $x_j \in [|a|/2^n, |a|/2^{n-1})$ for some n and consequently

$$1/2 \leq x_i/x_j \leq 2.$$

From *From Erdős to Kiev* by Ross Honsberger, Dolciani Mathematical Exposition, 1995, pp. 164–166.

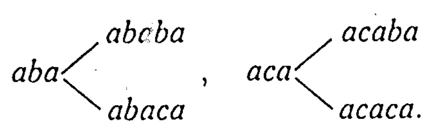
5. This problem concerns sequences $x_1 x_2 \cdots x_n$ in which each x_i is either a, b, or c. Determine the number of those sequences which have length n, begin and end with the letter a, and in which adjacent terms are always different letters.

Solution.

We begin by looking briefly at the sequences of a few small values of n. Let the number of sequences of length n be denoted by t_n.

n	Sequences	t_n
1	a	1
2	none	0
3	aba, aca	2
4	$abca$, $acba$	2
5	$ababa$, $abaca$, $acaba$, $acaca$, $abcba$, $acbca$	6

In attempting to see how sequences of length n might be derived from shorter ones, one might notice that a sequence of length n can be obtained by attaching either ba or ca at the end of any sequence of length $n-2$. For example,



Of course, all the sequences generated in this way have the letter a in the third-last position. Conversely, a sequence of length n whose third-last term is the letter a yields an acceptable sequence of length $n-2$ when its last two terms are dropped. Thus the number of sequences of length n in which the third-last term is the letter a is $2t_{n-2}$.

For the rest of the sequences of length n, the third-last term is either b or c, and each sequence of this kind provides a single sequence of length $n-1$ by simply deleting its second-last term:

$$a \cdots bca \to a \cdots ba,$$
$$a \cdots cba \to a \cdots ca.$$

(This is not allowed when the third-last term is the letter a.) Conversely, there is only one possible letter that can be inserted between the last two terms of a sequence, and doing so clearly extends one of length $n-1$ to one of length n. Therefore there are t_{n-1} sequences of length n in which the third-last term is b or c, and we have altogether that

$$t_n = t_{n-1} + 2t_{n-2}$$

with initial conditions $t_1 = 1$ and $t_2 = 0$. Now assume $t_n = r^n$. Substituting into the difference equation yields the characteristic equation $r^2 - r - 2 = 0$, from which $r = 2$ or $r = -1$. So the general solution is

$$t_n = c_1 2^n + c_2(-1)^n.$$

Using the initial conditions lets us find $c_1 = 1/6$ and $c_2 = -2/3$. Therefore,

$$t_n = \frac{1}{3}2^{n-1} + \frac{2}{3}(-1)^{n-1}.$$