1999 Missouri MAA Collegiate Mathematics Competition

Session I

1. Let $P \neq (0,0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at P will intersect the parabola at another point, say Q. Find the coordinates of P so that the length of the arc of the parabola between P and Q is a minimum.

Solution.

Let $P = (p, p^2)$ and $Q = (q, q^2)$. The slope of the tangent line to the parabola at P is $2p$ so the slope of the normal line to the parabola at P is $-1/2p$. Thus, the equation of the normal line to the parabola at P is

$$y - p^2 = -\frac{1}{2p}(x - p).$$

Hence, the other intersection point of the normal line and the parabola, Q has the x-coordinate

$$q = -p - \frac{1}{2p}.$$

Now the length of the arc of the parabola between P and Q is

$$L = \int_{-p-1/(2p)}^{p} \sqrt{1 + (2x)^2}\,dx.$$

Differentiating L with respect to p gives

$$\frac{dL}{dp} = \sqrt{1 + 4p^2} - \left(-1 + \frac{1}{2p^2}\right)\sqrt{1 + 4\left(-p - \frac{1}{2p}\right)^2}$$

and setting the derivative equal to 0 gives

$$\sqrt{1 + 4p^2} = \left(\frac{1}{2p^2} - 1\right)\sqrt{4p^2 + 5 + \frac{1}{p^2}}.$$

Squaring both sides to remove the radicals and multiplying by p^4, we obtain

$$12p^4 - p^2 - 1 = (4p^2 + 1)(3p^2 - 1) = 0.$$

Hence, the coordinates of p resulting in a minimum arc length are

$$\left(\pm\sqrt{\frac{1}{3}}, \frac{1}{3}\right).$$

2. Give a precise characterization of those points in the plane which do not lie on a tangent line to the curve $y = x^4 - 6x^2$.

Solution.

Given a point $(t, t^4 - 6t^2)$ on the curve, the equation of the tangent line at that point is $y = (4t^3 - 12t)x - (3t^4 - 6t^2)$. For a fixed value of x, as t varies, the maximum value of y will occur when

$$\frac{dy}{dt} = (12t^2 - 12)x - (12t^3 - 12t) = 0.$$

This polynomial factors as $(12t^2 - 12)(x - t)$, so the critical values occur at $t = 1, -1, x$. The corresponding y values are $-8x + 3$, $8x + 3$, and $x^4 - 6x^2$ and a point (x, y) will not lie on any tangent line if the y coordinate is greater than the maximum of these three quantities. More specifically, the set of points which lie on no tangent line to the curve is $\{(x, y) \mid |x| > 3 \text{ and } y > x^4 - 6x^2, \text{ or } |x| \leq 3, \text{ and } y > 8|x| + 3\}$.

Note that $y = \pm 8x + 3$ are the tangent lines at the points of inflection, $(\mp 1, -5)$, and that $(\pm 3, 27)$ are where these tangents intersect the quartic.

From *Scientific American*, June 1975, pp. 106–107.

3. On a 5×5 square matrix place 13 black counters and 12 white counters in alternating checkerboard fashion. Remove the black counter in the center square. Player A controls the white counters and B the black. They take turns moving one of their counters orthogonally to the vacant square until a player loses by being unable to move. Which player has a winning strategy? What is the strategy?

Solution.

This game was invented in the late 1960's by G. W. Lewthwaite of Scotland. Player B has a winning strategy. B's strategy is to imagine that the matrix, except for the center square, is covered by 12 nonoverlapping dominoes.

Whenever A moves, B simply moves his counter that is on the domino A has just vacated. Since this ensures that B always has a move to follow a move by A, B is sure to win in 12 or fewer moves. If the board is colored like a checkerboard, it is obvious that on each move a counter goes to a square of a different color and that no counter can be moved twice. The game therefore cannot go beyond 12 moves for each player.

Modified Proposal by Viktors Linis, University of Ottawa, Ottawa, Canada, Problem 162, *Crux Mathematicorum*, 2 (1976), 226–228.

From a list of problems submitted for the 1975 Canadian Mathematics Olympiad (but not used on the actual exam). This solution is adapted from solutions submitted independently by John L. Davison, Laurentian University, Sudbury, Ontario and Dan Eustice, Ohio State University, Columbus, Ohio.

4. If $x_0 = 5$ and $x_{n+1} = x_n + 1/x_n$, prove that for all $n \geq 1$

$$2n < x_n^2 - 25 < 47n/23.$$

Solution.

The proof of the result is by induction on n. The theorem is trivially true for $n = 1$ since $2 < (5.2)^2 - 25 < 47/23$. Assume it is true for $n = k$. Then, on the one hand

$$\begin{aligned} x_{k+1}^2 - 25 &= \left(x_k + \frac{1}{x_k}\right)^2 - 25 = (x_k^2 - 25) + 2 + \frac{1}{x_k^2} \\ &> 2k + 2 + \frac{1}{x_k^2} = 2(k+1) + \frac{1}{x_k^2} > 2(k+1), \end{aligned}$$

while on the other hand

$$\begin{aligned} x_{k+1}^2 - 25 = [(x_k^2 - 25) + 2] + \frac{1}{x_k^2} &< \left(\frac{47k}{23} + \frac{47}{23}\right) + \left(\frac{1}{x_k^2} - \frac{1}{23}\right) \\ &= \frac{47(k+1)}{23} - \left(\frac{x_k^2 - 23}{23x_k^2}\right) < \frac{47(k+1)}{23}. \end{aligned}$$

The last inequality follows because if $k \geq 1$, then $x_k > 5$ which implies $x_k^2 - 23 > 0$. Thus, the result is true for $n = k + 1$. Therefore, the induction proof is complete.

5. For an $n \times n$ matrix X, we say λ is an eigenvalue if $\det(\lambda I - X) = 0$. Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix. Prove that AB and BA have the same non-zero eigenvalues.

Solution.

Without loss of generality, we assume $m \geq n$. Let

$$X = \begin{pmatrix} A & 0 \end{pmatrix}_{m\times m} \text{ and } Y = \begin{pmatrix} B \\ 0 \end{pmatrix}_{m\times m}.$$

We are to show that

$$XY = AB \text{ and } YX = \begin{pmatrix} BA & 0 \\ 0 & 0 \end{pmatrix}$$

have the same eigenvalues.

For any positive ϵ small enough, $X + \epsilon I$ is non-singular. Therefore,

$$\begin{aligned}
\det(\lambda I - XY) &= \lim_{\epsilon\to 0} \det(\lambda I - (X + \epsilon I)Y) \\
&= \lim_{\epsilon\to 0} \det((X + \epsilon I)^{-1}) \det(\lambda I - (X + \epsilon I)Y) \det(X + \epsilon I) \\
&= \lim_{\epsilon\to 0} \det((X + \epsilon I)^{-1}(\lambda I - (X + \epsilon I)Y)(X + \epsilon I)) \\
&= \lim_{\epsilon\to 0} \det(\lambda I - Y(X + \epsilon I)) \\
&= \det(\lambda I - YX).
\end{aligned}$$

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Session II

1. Let SC be the semicircle with $y \geq 0$ centered at $(1,0)$ with radius 1. Let C_a be the circle with radius $a > 0$ and center $(0,0)$ and denote the point $(0,a)$ by P. Consider the line through P and the intersection of SC and C_a. What is the limiting position of the x-intercept of this line as $a \to 0$?

Solution.

The limiting position is $x = 4$. The circles intersect at

$$\left(\frac{a^2}{2}, \frac{a}{2}\sqrt{4-a^2}\right).$$

The line through this point and $(0,a)$ is

$$y = a + \left(\frac{\sqrt{4-a^2}-2}{a}\right)x$$

and the x-intercept is

$$x = \frac{a^2}{2-\sqrt{4-a^2}}.$$

Taking the limit as $a \to 0$ yields 4 as the limit.

2. Find the limit

$$\lim_{N\to\infty}\left(1-2\sum_{n=1}^{N}\frac{1}{16n^2-1}\right).$$

Solution.

Observe that

$$\frac{1}{16n^2-1}=\frac{1}{2}\left(\frac{1}{4n-1}-\frac{1}{4n+1}\right).$$

The expression in the parentheses above is thus

$$\sum_{n=0}^{2N}\frac{1}{2n+1}(-1)^n.$$

However, Gregory's series for the arctangent is

$$\mathrm{Tan}^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}.$$

The Alternating Series Test plus the Lagrange form of the remainder in Taylor's Theorem can be used to show that the series on the right converges at $x=1$ to the function on the left, and therefore to $\pi/4$.

3. For n positive real numbers with minimum m and maximum M, let A and G denote their arithmetic and geometric means. Prove that

$$A - G \geq n^{-1}(\sqrt{M} - \sqrt{m})^2.$$

Solution.

Let the n positive real numbers be $x_1, \ldots, x_n$ and assume $x_1 = m$ and $x_n = M$. The inequality

$$A - G \geq (\sqrt{M} - \sqrt{m})^2/n$$

is equivalent to

$$\sum_{j=1}^{n} x_j - n\left(\prod_{j=1}^{n} x_j\right)^{1/n} \geq M - 2\sqrt{Mm} + m$$

or

$$\sum_{j=2}^{n-1} x_j + 2\sqrt{Mm} \geq n\left(\prod_{j=1}^{n} x_j\right)^{1/n}.$$

Applying the AM-GM inequality to the n numbers $\sqrt{Mm}, x_2, x_3, \ldots, x_{n-1}, \sqrt{Mm}$, we get

$$\sum_{j=2}^{n-1} x_j + 2\sqrt{Mm} \geq n \cdot \left(\prod_{j=2}^{n-1} x_j \cdot \sqrt{Mm}\sqrt{Mm}\right)^{1/n} = n \cdot \left(\prod_{j=1}^{n} x_j\right)^{1/n},$$

as needed.

4. Find all possible continuous and differentiable curves C which have the following properties. The curve C lies in the first quadrant and contains the point $(0,0)$. Whenever P is on C the interior of the rectangle R bounded by the coordinate axes and horizontal and vertical lines through P is separated into two parts by C. When the part adjacent to the x-axis is rotated about the x-axis and the part adjacent to the y-axis is rotated about the y-axis, two solids of equal volume are generated.

Solution.

In order for C to always divide R into two and only two separate parts, C must be an increasing function. Let C be given by $y = f(x)$. The volumes of the two solids are

$$V_1 = \int_0^x \pi(f(t))^2 dt \quad \text{and} \quad V_2 = \int_0^x 2\pi t(f(x) - f(t))dt.$$

Setting $V_1 = V_2$, canceling the π and breaking the V_2 integral into two parts gives

$$\int_0^x (f(t))^2 dt = \int_0^x 2tf(x)dt - \int_0^x 2tf(t)dt.$$

Differentiate both sides with respect to x and simplify to get

$$(f(x))^2 = x^2 f'(x) \quad \text{or} \quad y^2 = x^2 \frac{dy}{dx}.$$

Separation of variables and integration gives the family of curves

$$y = \frac{x}{1 + cx}.$$

The point $(0,0)$ is on each curve in the family, and if $c \geq 0$ the domain is $[0, \infty)$, while if $c < 0$ the domain is $[0, 1/|c|)$.

5. Let A_n denote the $n \times n$ matrix whose (i,j) entry is $\mathrm{GCD}(i,j)$. Compute $\det(A_n)$.

Solution.

We will show that one can use elementary column operations to reduce $A_n = (v_1, v_2, \ldots, v_n)$, where the v_j is the infinite vector whose ith coordinate is $\mathrm{GCD}(i,j)$, to $(w_1, w_2, \ldots, w_n)$, where w_j is the vector whose ith coordinate is $\phi(j)$ if $j|i$ and 0 otherwise, $\phi(j)$ being the Euler totient function which counts the number of positive integers less than or equal to j and relatively prime to it. First we need a lemma.

Lemma.

$$\sum_{i|n} \phi(i) = n.$$

Proof. The number of elements of order i in the group $\mathbb{Z}_n$ is $\phi(i)$ if $i|n$ and 0 otherwise. The expression on the left is the number of elements in $\mathbb{Z}_n$ grouped by order, but this is clearly n.

We prove the result by induction. The case $n = 1$ is clear. Given $(v_1, v_2, \ldots, v_n, v_{n+1})$ as above, by the inductive hypothesis we can use elementary column operations to reduce this to $(w_1, w_2, \ldots, w_n, v_{n+1})$. Consider

$$\sum_{j|n+1} w_j.$$

Its ith coordinate is

$$\sum_{\substack{j|n+1 \\ j|i}} \phi(j) = \sum_{j|\mathrm{GCD}(n+1,i)} \phi(j) = \mathrm{GCD}(n+1, i),$$

so

$$\sum_{j|n+1} w_j = v_{n+1}.$$

Therefore,

$$v_{n+1} - \sum_{\substack{j|n+1 \\ j \neq n+1}} w_j = w_{n+1}$$

and we have column-reduced our original matrix to $(w_1, w_2, \dots, w_n, w_{n+1})$.

We can use the column operations above to reduce the matrix A_n to an upper triangular matrix with $\phi(j)$ on the diagonal $j = 1, 2, \dots, n$, so

$$\det(A_n) = \prod_{j=1}^{n} \phi(j).$$