1997 Missouri MAA Collegiate Mathematics Competition

Session I

1. Let $P \neq (0,0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at P will intersect the parabola at another point, say Q. Find the coordinates of P so that the y-coordinate of Q is a minimum.

Solution.

Let P have the coordinates (p, p^2) and Q have the coordinates (q, q^2). Without loss of generality, let $p > 0$. The slope of the tangent line to the parabola at P is $2p$ so the slope of the normal line to the parabola at P is $-1/2p$. Thus, the equation of the normal line to the parabola at P is

$$y - p^2 = \frac{-1}{2p}(x - p).$$

Therefore, the point Q, the intersection of the normal line and the parabola has the x-coordinate

$$q = -p - \frac{1}{2p}.$$

Hence, the y-coordinate of Q is

$$y = q^2 = \left(-p - \frac{1}{2p}\right)^2 = p^2 + 1 + \frac{1}{4p^2}.$$

Taking the derivative of y as a function of p,

$$y' = 2p - \frac{1}{2}p^{-3} = \frac{4p^4 - 1}{2p^3}.$$

Minimizing y as a function of p yields $p = \sqrt{1/2}$ so

$$P = \left(\sqrt{\frac{1}{2}}, \frac{1}{2}\right).$$

From Crux Mathematicorum, 1979, Practice 3-2

2. Prove that from any row of n integers one may always select a block of adjacent integers whose sum is divisible by n.

Solution.

Let the row of integers be $a_1, a_2, \ldots, a_n$ and consider the n sums

$$s_1 = a_1,\ s_2 = a_1 + a_2,\ \ldots\ , s_n = a_1 + a_2 + \cdots + a_n.$$

If some $s_i \equiv 0 \pmod{n}$, we are through. Otherwise we have n sums with at most $n-1$ possible residues modulo n; hence, by the pigeonhole principle, there must be two sums s_j and s_k, with $j > k$, such that $s_j \equiv s_k \pmod{n}$, and then

$$s_j - s_k = a_{k+1} + a_{k+2} + \cdots + a_j \equiv 0 \pmod{n}.$$

3. Find conditions on the parameters a, b, c, and d so that

$$f(x, y) = a \sin(x + y) + b \cos(x + y) + c \sin(x - y) + d \cos(x - y)$$

can be written as $f(x, y) = g(x)h(y)$.

Solution.

Expansion of the sines and cosines gives

$$f(x, y) = (a + c) \sin x \cos y + (a - c) \cos x \sin y + (b + d) \cos x \cos y + (d - b) \sin x \sin y.$$

So, if the factorization is possible,

$$f(x, y) = (A \sin x + B \cos x)(C \sin y + D \cos y),$$

or

$$f(x, y) = AD \sin x \cos y + BC \cos x \sin y + BD \cos x \cos y + AC \sin x \sin y.$$

Thus we need to have

$$\begin{aligned} a + c &= AD \\ a - c &= BC \\ b + d &= BD \\ d - b &= AC. \end{aligned}$$

Hence,

$$\frac{a+c}{d-b} = \frac{AD}{AC} = \frac{D}{C} = \frac{BD}{BC} = \frac{b+d}{a-c}.$$

Therefore,

$$a^2 + b^2 = c^2 + d^2.$$

4. A point P is in the interior of a circle of radius r. Place the vertex of a right angle at P and denote by A and B the points where the sides of the right angle intersect the circle. Let Q be the point which completes the rectangle $PAQB$. What is the locus of Q?

Solution.

Let the center of the circle be the origin, O, and let $\overrightarrow{p}$ be the vector from O to P. $|\overrightarrow{p}|^2 < r^2$ since P is in the interior of the circle. Let $\overrightarrow{v} = \overrightarrow{PA}$ and $\overrightarrow{w} = \overrightarrow{PB}$. We know $\overrightarrow{v} \perp \overrightarrow{w}$. Let $\overrightarrow{a} = \overrightarrow{OA}$ and $\overrightarrow{b} = \overrightarrow{OB}$, and note that $|\overrightarrow{a}| = |\overrightarrow{b}| = r$. Also, $\overrightarrow{a} = \overrightarrow{p} + \overrightarrow{v}$ and $\overrightarrow{b} = \overrightarrow{p} + \overrightarrow{w}$. If $\overrightarrow{q} = \overrightarrow{OQ} = \overrightarrow{p} + \overrightarrow{v} + \overrightarrow{w}$, then

$$\begin{aligned}
|\overrightarrow{q}|^2 &= (\overrightarrow{p} + \overrightarrow{v} + \overrightarrow{w}) \cdot (\overrightarrow{p} + \overrightarrow{v} + \overrightarrow{w}) \\
&= |\overrightarrow{p}|^2 + 2\overrightarrow{p} \cdot \overrightarrow{v} + 2\overrightarrow{p} \cdot \overrightarrow{w} + \overrightarrow{v} \cdot \overrightarrow{v} + \overrightarrow{w} \cdot \overrightarrow{w} \\
&= |\overrightarrow{p}|^2 + \overrightarrow{a} \cdot \overrightarrow{v} + \overrightarrow{b} \cdot \overrightarrow{w} + \overrightarrow{p} \cdot \overrightarrow{v} + \overrightarrow{p} \cdot \overrightarrow{w} \\
&= |\overrightarrow{p}|^2 + (\overrightarrow{a} + \overrightarrow{p}) \cdot \overrightarrow{v} + (\overrightarrow{b} + \overrightarrow{p}) \cdot \overrightarrow{w} \\
&= |\overrightarrow{p}|^2 + (\overrightarrow{a} + \overrightarrow{p}) \cdot (\overrightarrow{a} - \overrightarrow{p}) + (\overrightarrow{b} + \overrightarrow{p}) \cdot (\overrightarrow{b} - \overrightarrow{p}) \\
&= |\overrightarrow{p}|^2 + |\overrightarrow{a}|^2 - |\overrightarrow{p}|^2 + |\overrightarrow{b}|^2 - |\overrightarrow{p}|^2 \\
&= 2r^2 - |\overrightarrow{p}|^2.
\end{aligned}$$

Thus, the distance from O to Q is constant, and the locus of Q is a circle centered at O with $(\text{radius})^2$ $2r^2 - |\overrightarrow{p}|^2$.

5. Let $\{L_n\}_{n=0}^{\infty}$ be the sequence of Lucas numbers: $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Let $DR(N)$ denote the digital root of a positive integer N, defined as the sum of the digits of N, composed enough times until a value between 1 and 9 is obtained. For example, $DR(667) = DR(19) = DR(10) = 1$. Show that there is a smallest positive integer k such that $DR(L_{n+k}) = DR(L_n)$ for all integers $n \geq 0$.

Solution.

Let the integer $N > 0$ be represented by

$$N = \sum_{i=0}^{r} a_i 10^i,$$

where the integers a_i satisfy $1 \leq a_r \leq 9$, $0 \leq a_i \leq 9$ for $i = 0, 1, 2, \ldots, r-1$, and $a_i = 0$ for $i > r$. Then

$$N = \sum_{i=0}^{r} a_i + \sum_{i=1}^{r} a_i (10^i - 1)$$

and as $9 \mid (10^i - 1)$, then

$$N \equiv \sum_{i=0}^{r} a_i \pmod 9,$$

where

$$\sum_{i=0}^{r} a_i = DS(N)$$

is the digital sum of N. Continuation of this reasoning with $DS(N)$ itself leads ultimately to $N \equiv DR(N) \pmod 9$. It follows that if N, M are two positive integers, then

$$N + M \equiv DR(N) + DR(M) \pmod 9.$$

But

$$N + M \equiv DR(N + M) \pmod 9,$$

so

$$DR(N + M) \equiv DR(N) + DR(M) \equiv DR[DR(N) + DR(M)] \pmod 9.$$

As digital roots lie in $[1, 9]$, then we must have the equality

$$DR(N + M) = DR[DR(N) + DR(M)].$$

Since

$$L_n = L_{n-1} + L_{n-2} \quad (n \geq 2),$$

then the sequence of digital roots of the Lucas numbers is completely determined by the first two: $DR(L_0) = 2$, $DR(L_1) = 1$. There are only 81 distinct pairs of the integers $1, 2, \ldots 9$, so there must be some minimal starting integer s and some minimal period integer p such that $DR(L_{s+p}) = DR(L_s)$ and $DR(L_{s+p+1}) = DR(L_{s+1})$. By inspecting the Lucas sequence, it follows that $s = 0$ and $p = 24$. Therefore, the smallest positive integer k such that

$$DR(L_{n+k}) = DR(L_n)$$

for all integers $n \geq 0$ is 24.

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Session II

Modified from the 1979 Putnam Exam.

1. Find positive integers n and $a_1, a_2, \dots, a_n$ such that

$$a_1 + a_2 + \cdots + a_n = 1997$$

and the product $a_1 a_2 \cdots a_n$ is as large as possible.

Solution.

We see that $n = 666$ and that all but one of the a_i equal 3 and the exceptional a_i is a 2 as follows. No a_i can be greater than 4 since one could increase the product by replacing 5 by $2 \cdot 3$, 6 by $3 \cdot 3$, 7 by $3 \cdot 4$, etc. There cannot be both a 2 and a 4 or three 2's among the a_i since $2 \cdot 4 < 3 \cdot 3$ and $2 \cdot 2 \cdot 2 < 3 \cdot 3$. Also, there cannot be two 4's since $4 \cdot 4 < 2 \cdot 3 \cdot 3$. Clearly, no a_i is a 1. Hence the a_i are 3's except possibly for a 4 or for a 2 or for two 2's. Since $1997 = 3 \cdot 665 + 2$, the only exception is a 2 and $n = 666$.

From the All-Soviet Mathematics Competition, 1962 - Problem 19

2. Let a, b, c, d be positive numbers with $abcd = 1$. Prove that

$$a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd \geq 10.$$

Solution.

By the A.M.-G.M. $\neq$,

$$\frac{a^2 + b^2 + c^2 + d^2}{4} \geq \sqrt[4]{a^2 b^2 c^2 d^2} = 1$$

and

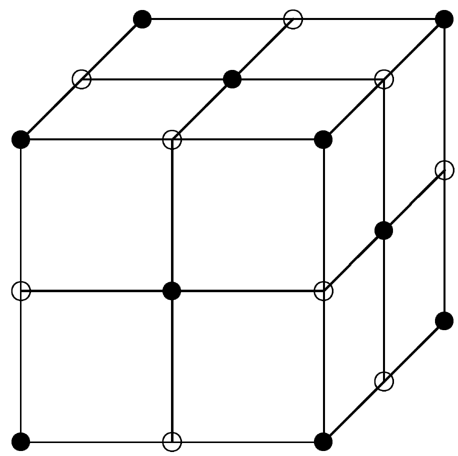
$$\frac{ab + ac + ad + bc + bd + cd}{6} \geq \sqrt[6]{a^3 b^3 c^3 d^3} = 1.$$

The result follows.

From one of Martin Gardner's books

3. A wooden cube of edge 3 is formed by gluing together 27 small cubes of edge 1. A termite, beginning with any one of the outer small cubes, begins to eat its way through the large cube, always moving perpendicular to a face (i.e., no diagonal movements are allowed - don't ask why, who knows the mind of a termite?) Is it possible for the termite to follow a path entirely within the large cube (emerging and crawling on the outside is also not allowed) which passes through each small cube exactly once and ends in the center cube? Generalize the problem to the case where the large cube has edge n, an odd integer.

Solution.



Consider the graph with a vertex for each small cube and an edge connects two vertices if their corresponding cubes are adjacent.

Now, suppose the termite begins at an outer vertex, follows the edges of the graph, visits each vertex exactly once, and ends at the center vertex. This path alternates visiting the 13 $\circ$ and 14 $\bullet$ vertices and must end in the center $\circ$ vertex. However, no matter if we start at an outer $\circ$ or $\bullet$ vertex, this is impossible.

In the $5 \times 5 \times 5$ case, if the corner vertex is a $\bullet$ vertex, then there are 62 $\circ$ vertices, 63 $\bullet$ vertices and the center vertex is a $\bullet$ vertex. By a parity argument, the only possible requested paths, if they exist, must start in an outer $\bullet$ vertex. Furthermore, by experimentation, if the termite starts in any $\bullet$ vertex on the outside, then the termite can travel on a path which visits every vertex exactly once and ends in the center $\bullet$ vertex.

Similar arguments show that if $n = 2k-1$ and k is even, then the termite cannot start on an outer vertex and end up in the center. But, if $n = 2k-1$ and k is odd, then the termite can start in a corner vertex and end up in the center.

From an article by Nancy Edwards in *The Pentagon*, Fall 1971.

4. Define a family of curves by

$$S_n = \{(x, y) : y = \frac{1}{n}\sin(n^2 x),\ 0 \le x \le \pi\},$$

where n is a positive integer. What is the limit of the length of S_n as $n \to \infty$?

Solution.

Denote the length of S_n by $L(S_n)$. Then

$$L(S_n) = \int_0^\pi \sqrt{1 + n^2\cos^2(n^2 x)}dx.$$

Clearly,

$$\sqrt{1 + n^2\cos^2(n^2 x)} > \sqrt{n^2\cos^2(n^2 x)} = n|\cos(n^2 x)|,$$

so

$$L(S_n) > \int_0^\pi n|\cos(n^2 x)|dx.$$

The period of $\cos(n^2 x)$ is $2\pi/n^2$, and $\cos(n^2 x) \ge 0$ for $0 \le x \le \pi/2n^2$, so

$$\begin{aligned} L(S_n) &> 2n^3 \int_0^{\pi/2n^2} \cos(n^2 x)dx \\ &= 2n^3 \left(\frac{\sin(n^2 x)}{n^2}\right)_0^{\pi/2n^2} \\ &= 2n. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} L(S_n) = \infty.$$

From the 1982/3 International Mathematical Olympiad

5. Consider the infinite sequences $\{x_n\}$ of positive real numbers with the following properties:

$$x_0 = 1, \quad \text{and for all } i \geq 0, \quad x_{i+1} \leq x_i.$$

(a) Prove that for every such sequence, there is an $n \geq 1$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

(b) Find such a sequence for which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} < 4 \text{ for all } n.$$

Solution.

(a) We will prove the series

$$(1) \qquad \frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \cdots, \quad \text{where } 1 = x_0 \geq x_1 \geq x_2 \geq \cdots > 0,$$

has sum ≥ 4 (with the obvious convention that this holds if the series diverges). This clearly implies that some partial sum of the series is ≥ 3.999.

Let L be the inf (= greatest lower bound) of the sums of all series of the form (1). Clearly $L > 1$ since the first term $\frac{1}{x_1} \geq 1$. For any $\epsilon > 0$, we can find a sequence $\{x_n\}$ such that

$$(2) \qquad L + \epsilon > \frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \cdots.$$

Setting $y_n = x_{n+1}/x_1$ $(n \geq 0)$, we note that $1 = y_0 \geq y_1 \geq y_2 \geq \cdots > 0$. The series on the right side of (2) can be written in the form

$$\frac{1}{x_1} + x_1\left(\frac{y_0^2}{y_1} + \frac{y_1^2}{y_2} + \frac{y_2^2}{y_3} + \cdots\right).$$

By the definition of L, the series in parentheses has sum $\geq L$. Hence from (2) we have

$$L + \epsilon > \frac{1}{x_1} + x_1 L.$$

Applying the A.M.-G.M. inequality to the right side, we get $L + \epsilon > 2\sqrt{L}$. Since this holds for all $\epsilon > 0$, it follows that $L \geq 2\sqrt{L}$. Hence $L^2 \geq 4L$, and since $L > 0$, this implies that $L \geq 4$.

(b) Let $x_n = 1/2^n$. Then

$$\sum_{n=0}^{\infty} \frac{x_n^2}{x_{n+1}} = \sum_{n=0}^{\infty} \frac{1}{2^{n-1}} = 4,$$

so all partial sums of the series are < 4.