2018 Missouri Collegiate Mathematics Competition
Session I

1. Let f be a real-valued function such that for any real x,

(a) $f(10+x) = f(10-x)$.

(b) $f(20+x) = -f(20-x)$.

Prove that f is an odd and periodic function.

2. Let $a > 0$. Suppose f is a continuous function on $[0, a]$ such that $f(x)\, f(a-x) = 1$. Evaluate

$$\int_0^a \frac{dx}{1+f(x)}.$$

3. In a theoretical version of the Canadian lottery "Lotto 6–49", a ticket consists of six distinct integers chosen by the player from 1 to 49 (inclusive). A t-prize is awarded for any ticket having t or more numbers in common with a designated "winning" ticket. Denote by $f(t)$ the smallest number of tickets required to be certain of winning a t-prize. Clearly $f(1) = 8$ and $f(6) = \binom{49}{6}$. Show that $f(2) \leq 19$.

4. Find all real numbers x that satisfy $6^x + 1 = 8^x - 27^{x-1}$. Prove your solution set contains all solutions.

5. Let P_1, P_2, and P_3 be three points on the parabola $y = x^2$, and let ℓ_1, ℓ_2, and ℓ_3 be the tangent lines to the parabola at these points. The tangent lines intersect pairwise in three points. Denote the intersection of ℓ_1 and ℓ_2 by Q_{12}, the intersection of ℓ_1 and ℓ_3 by Q_{13}, and the intersection of ℓ_2 and ℓ_3 by Q_{23}. Find the ratio of the area of triangle $P_1P_2P_3$ to the area of triangle $Q_{12}Q_{13}Q_{23}$.

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Session II

1. For each positive integer n, let $a_n = \sum_{k=0}^{n} \frac{\pi^{4k}}{(4k+1)!}$ and $b_n = \sum_{k=0}^{n} \frac{\pi^{4k}}{(4k+3)!}$. Evaluate $\lim_{n\to\infty} \frac{a_n}{b_n}$.

2. Triangle ABC is a right triangle with right angle at C. Set $a = BC$, $b = CA$, and $c = AB$. Inscribe a circle inside triangle ABC, and inside the circle inscribe a triangle similar to triangle ABC. Repeat the process, obtaining an infinite sequence of circles and similar triangles. Find in terms of a, b, c the sum of the areas of the circles.

3. What is the coefficient of x^{2018} in

$$(x+1)^3(x^2+1)^4(x^4+1)^5 \times (x^8+1)(x^{16}+1)(x^{32}+1)(x^{64}+1)(x^{128}+1)(x^{256}+1)(x^{512}+1)(x^{1024}+1)?$$

4. Let P_2 denote the set of real polynomials of degree less than or equal to 2. Define the map $J : P_2 \to \mathbb{R}$ by

$$J(f) = \int_0^1 [f(x)]^2 \, dx.$$

Let $Q = \{f \in P_2 : f(1) = 1\}$. Show that J attains a minimum value on Q, and determine where the minimum occurs.

5. If A, B, C are angles of an acute triangle, prove that

$$(\tan A + \tan B + \tan C)^2 \geq (\sec A + 1)^2 + (\sec B + 1)^2 + (\sec C + 1)^2.$$