2017 Missouri Collegiate Mathematics Competition
Session I

1. An *autobiographical number* is a natural number with ten digits or less in which the first digit of the number (reading from left to right) tells you how many zeros are in the number, the second digit tells how many 1's, the third digit tells you how many 2's, and so on. For example,

$$6,210,001,000$$

is an autobiographical number. Find the smallest autobiographical number and prove that it is the smallest.

2. Recall that the trace $\mathrm{tr}(A)$ of an $n \times n$ matrix A is the sum of the entries on the main diagonal of A. Let A^T denote the transpose of the matrix A. Show that if A and B are $n \times n$ real-valued matrices with $\mathrm{tr}(AA^T + BB^T) = \mathrm{tr}(AB + A^T B^T)$, then it must be the case that $A = B^T$.

3. Let f be a real-valued continuous function on the closed interval $[0, 1]$. Assuming that $\int_0^1 f(x)\,dx = \frac{\pi}{4}$, show that there is some y with $0 < y < 1$ and $\frac{1}{1+y} < f(y) < \frac{1}{2y}$.

4. A triangle ABC has positive integer sides. Suppose $A = 2B$, and $C > 90°$. Find the minimum length of the perimeter of ABC.

5. Suppose that x and y are chosen at random from the uniform distribution on the square $(0, 2] \times (0, 2]$. Given that $(x, y, 1)$ are the sides of a triangle, find the probability that the triangle is acute.

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Session II

1. A smooth function $f(x)$ has $f''(x) > 0$ for all x in $[0, 1]$. For each point a in $[0, 1]$, draw the tangent line to $y = f(x)$ at the point where $x = a$. Let $A(a)$ be the area bounded by the curve $y = f(x)$, the tangent line at a, and the lines $x = 0$ and $x = 1$. What value(s) of a minimizes the area?

2. Let n be a positive integer. Evaluate

$$\int_0^{\pi/2} \frac{\sin\left[(2n+1)\,x\right]}{\sin x}\,dx.$$

3. Let n be a fixed positive integer. Let $M_n\left(\mathbb{Z}_2\right)$ denote the set of $n \times n$ matrices with entries in $\mathbb{Z}_2$. Given $A \in M_n\left(\mathbb{Z}_2\right)$, what is the probability that A is invertible? (Hint: Recall that $\mathbb{Z}_2 = \{0, 1\}$ and all arithmetic is modulo 2.)

4. Find, with proof, all solutions in nonnegative integers x, y, z to

$$2^x + 4^y = 6^z. \tag{1}$$

5. Given 2017 points in the plane, what is the largest number of line segments that can be drawn connecting these points such that distinct segments intersect only at their endpoints?