2016 Missouri Collegiate Mathematics Competition
Session I

1. Let $\mathcal{R}$ be the region in the first quadrant bounded by the x-axis and the graph of $y = \sin x$ from $x = 0$ to $x = \pi$. The graph of $y = \cos x$ divides $\mathcal{R}$ into two regions. Let $\mathcal{R}_1$ be the subregion of $\mathcal{R}$ bounded on the left by $y = \cos x$, on the right by $y = \sin x$, and below by the x-axis. Find a positive number b such that the line $y = b$ divides $\mathcal{R}_1$ into two regions of equal area.

2. Find, with proof, the sum of the series

$$\sum_{n=1}^{\infty} \sin \frac{1}{2^{n+1}} \cos \frac{3}{2^{n+1}}.$$

3. Find, with proof, all positive integers x satisfying $3^{(2^{x!})} = 2^{(3^{x!})} + 1$.

4. Find a polynomial with integer coefficients that has $2^{\frac{1}{5}} + 2^{-\frac{1}{5}}$ as a root.

5. For each positive integer n, let

$$\mathcal{R}_n = \left\{(x, y) : 0 \leq x \leq n \text{ and } 0 \leq y \leq \sqrt{x}\right\}.$$

Let $N(n)$ denote the number of points in $\mathcal{R}_n$ whose coordinates are both integers. Compute

$$\lim_{n \to \infty} \frac{N(n)}{n^{3/2}}.$$

2016 Missouri Collegiate Mathematics Competition
Session II

1. Evaluate

$$\sum_{k=1}^{n} k2^{k-1}$$

for $n = 1, 2, 3, \ldots$.

2. Evaluate

$$\int_1^{\infty} \frac{\lfloor x \rfloor}{x^3}\, dx,$$

where $\lfloor x \rfloor$ denotes the greatest integer function.

3. Find the smallest positive angle θ in degrees satisfying

$$\sin^2(2015\theta) + \cos^2(2016\theta) = 1.$$

4. If a and b are the radii of two spheres, tangent to each other and to a plane (on the same side of the plane), the radius x of the largest sphere which can pass between them is given by what formula?

5. In a regular octagon each side is colored blue or yellow. From such a coloring, another coloring will be obtained "in one step" as follows: if the two neighbors of a side have different colors, the "new" color of the side will be blue, otherwise the color will be yellow. (Note that the colors are modified simultaneously.) Show that after a finite number, say N, of moves, all sides will be colored yellow. What is the least value of N that works for all possible colorings?