2015 Missouri Collegiate Mathematics Competition
Session I

1. Find numbers a and b such that

$$\lim_{x \to 0} \frac{\sqrt{ax+b}-2}{x} = 1.$$

2. If p, q, and r are distinct roots of $x^3 - x^2 + x - 2 = 0$, find the value of $p^3 + q^3 + r^3$.

3. Show that for each positive integer n, the function $f_n(x) = x^n + (x-1)^n - (x+1)^n$ has a unique nonzero real root r_n and that $r_n \leq r_{n+1}$ for all n.

4. Consider the set S of all integer-valued triples (x, y, z) satisfying $6x+10y-15z = 1$. Find all such triples satisfying $0 \leq x, y, z \leq 2015$ and identify all such triples which give the maximum possible value of $y + z$.

5. For each positive integer n let $g(n)$ be the number of digits greater than 4 in the decimal expansion of 2^n. Is it true that

$$\sum_{n=1}^{\infty} \frac{g(n)}{2^n} = \frac{2}{9}?$$

Prove or disprove.

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Session II

1. A smooth curve crosses the y-axis at the point $(0,4)$ and is such that given any point P on the curve, the tangent line to the curve at P crosses the x-axis at a point Q whose x-coordinate is 2015 more than the x-coordinate of P. Determine the area of the region in the first quadrant bounded by the x-axis, the y-axis, and this curve.

2. Consider the following two-person game: Start with 2015 pennies on a table. Players A and B alternate turns, with A going first. A legal move consists of removing any divisor of the number of pennies on the table, as long as the divisor is *strictly less than* the number of pennies on the table. For example, at the start of the game, player A can remove 1, 5, 13, 31, 65, 155, or 403 pennies, but not 2015 pennies. If player A removes 5 pennies, then player B could remove, say, 3 pennies, etc.

The game ends when no legal move is possible (i.e., when only one penny remains), and whoever's turn it is loses (and the opponent wins). So the objective is to leave your opponent with just one penny. Is there a winning strategy? If so, who wins, A or B?

3. A certain game of chance involves choosing 20 distinct integers from the set $\{1, 2, \ldots, 80\}$. While waiting to see if this selection would hypothetically lead to winning a huge sum of money, you wonder if there always exist two nonempty and disjoint subsets of the chosen 20 integers having equal sums of squares of their elements. Show that, yes, two such subsets always exist.

4. Prove that the sum of any finite subsequence of consecutive terms of the harmonic sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ written in lowest terms has an odd numerator.

5. Let $x_1 = 1$ and

$$x_{n+1} = \frac{1}{x_n}\left(\sqrt{1 + x_n^2} - 1\right)$$

for $n \geq 1$. Show that the sequence $\{2^n x_n\}$ converges and finds its limit.