2011 Missouri Collegiate Mathematics Competition
Session I

1. Consider the parabola $x^2 = 4py$, with vertex at the origin, focus at $F(0, p)$, and directrix $y = -p$. Let $P(x, y)$ be a point on the parabola and let $R(x, -p)$ be the intersection of the vertical line through P with the directrix. Determine P so that the triangle PFR is equilateral.

2. Prove that the inequality $\sin x + \arcsin x > 2x$ holds for all values of x such that $0 < x \leq 1$.

3. For what non-negative integers n does there exist a polynomial P_n of degree n with integer coefficients satisfying $P_n(k) = 2^k$ for all integers k, $0 \leq k \leq n$? Find all such polynomials.

4. Let a and b be real numbers with $b \neq 0$ and let $C(\mathbb{R})$ denote the set of continuous functions from the reals to the reals. Define $T : C(\mathbb{R}) \to C(\mathbb{R})$ by $(Tf)(x) = ax + b\int_0^x f(t)\,dt$. Find all fixed points of T^2, where T^2 denotes the composition of T with itself.

5. Let the function F be given by

$$F(x) = e^{-x} - \left(1 - \frac{x}{n}\right)^n.$$

Show that, for $n \geq 2$,

$$0 \leq F(x) \leq \frac{e^{-1}}{n} \qquad \text{on} \qquad [0, n].$$

2011 Missouri Collegiate Mathematics Competition
Session II

1. For $0 < r < 1$, let n_r denote the line that is normal to the curve $y = x^r$ at the point $(1, 1)$, and let S_r denote the region in the first quadrant of the xy-plane bounded by the x-axis, the curve $y = x^r$, and the line n_r. Find the value of r that minimizes the area of S_r.

2. A rectangular strip is divided into n bands of equal width as in the following figure.

1	2						n

Each band is colored by one of m colors. Two patterns are considered identical if one is a (left-to-right) mirror reflection of the other. Determine the number of distinct patterns with m colors and n bands.

3. Let M be an arbitrary 3×3 matrix, each of whose entries m_{ij} is drawn from the set $\{-1, 1\}$, and let $\det M$ be the determinant of M. Prove that $|\det M| \leq 4$.

4. Calculate

$$\int_0^2 \left(\sqrt{1 + x^3} + \sqrt[3]{x^2 + 2x} \right) dx.$$

5. Evaluate the series

$$\sum_{n=0}^{\infty} \frac{1}{2011^{2^n} - 2011^{-2^n}} = \frac{1}{2011^1 - 2011^{-1}} + \frac{1}{2011^2 - 2011^{-2}} + \frac{1}{2011^4 - 2011^{-4}} + \dots$$

and express it as a rational number.