2010 Missouri Collegiate Mathematics Competition

Session I

1. For the parabola having equation $y = -x^2$ let $a < 0$ and $b > 0$ with points $P : (a, -a^2)$ and $Q : (b, -b^2)$. Let M be the midpoint of PQ and let R be the intersection of the vertical line through M with the parabola. Show that the area of the region bounded by the parabola and the line segment PQ is $\frac{4}{3}$ of the area of triangle PQR. (Archimedes, 3rd century B.C.)

2. Suppose that the length of the larger base of an isosceles trapezoid equals the length of a diagonal and the length of the smaller base equals the altitude. Find the ratio of the length of the larger base to the length of the smaller base.

3. Consider the Diophantine equation $x(2x^2 + 3x + 3) = y^3 - 1$. Prove that it does not have a solution (x, y) in positive integers, or find such a solution if it does.

4. The Fibonacci and Lucas numbers are sequences defined by the following initial conditions and second order recurrence relations. Let $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$ define $F_n = F_{n-1} + F_{n-2}$; let $L_0 = 2$, $L_1 = 1$, and for $n \geq 2$ define $L_n = L_{n-1} + L_{n-2}$.

Let $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. The Binet form of the Fibonacci and Lucas numbers are given by

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where n is any nonnegative integer.

Let k be a fixed positive integer and define

$$U_n = \frac{F_{kn}}{L_k},$$

where n is any nonnegative integer. Find U_0 and U_1 and find a second order recurrence relation involving L_k satisfied by the sequence $\{U_n\}_{n=0}^{\infty}$.

5. If your calculator is set to radian mode, and you enter any number and then repeatedly push the "cosine" button, the displayed value will converge to $0.73908\ldots$. Call this number d. If $f(x) = x - \cos x$, then $f(d) = 0$. The number d can be expressed as a series in odd powers of π:

$$d = \sum_{n=0}^{\infty} a_n \pi^{2n+1}.$$

Find a_0 and a_1.

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Session II

1. Let S be the collection of ordered pairs (x, y) in $[0,1] \times [0,1]$ such that either x or y is irrational. Prove or disprove that for any two distinct ordered pairs in S, we can find a path in S connecting the two points.

2. If a, b, c are positive real numbers, find the value of x that minimizes the function

$$f(x) = \sqrt{a^2 + x^2} + \sqrt{(b-x)^2 + c^2}.$$

(Hint: Think geometrically.)

3. A sequence of 2×2 matrices, $\{M_n\}_{n=1}^{\infty}$, is defined as follows:

$$M_n = \begin{pmatrix} m_{11} = \frac{1}{(2n+1)!} & m_{12} = \frac{1}{(2n+2)!} \\ m_{21} = \sum_{k=0}^{n} \frac{(2n+2)!}{(2k+2)!} & m_{22} = \sum_{k=0}^{n} \frac{(2n+1)!}{(2k+1)!} \end{pmatrix}.$$

For each n, let $\det M_n$ denote the determinant of M_n. Determine the value of

$$\lim_{n\to\infty} \det M_n.$$

4. Evaluate the integral

$$I = \int_{\frac{1}{2}}^{2} \frac{\ln x}{1+x^2} dx.$$

5. A function f has the following properties:

(a) For all $x \geq 1$, $f(x)$ is a positive, differentiable, decreasing function;
(b) Whenever x equals a natural number k, we set $f(k) = f_k$, an element of a numerical sequence;
(c) The series $\sum_{k=1}^{\infty} f_k$ diverges to ∞;
(d) F is an arbitrary antiderivative of f, but with a fixed constant of integration, and is defined for all $x \geq 1$;
(e) $\lim_{x\to\infty} F(x) = \infty$.

Prove that

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} f_k - F(n)\right)$$

is finite.