2008 Missouri Collegiate Mathematics Competition

Session I

1. A straight line segment of arbitrary positive slope m is drawn from the origin O to the point of intersection A in quadrant I with the ellipse $x^2 + 4y^2 = 4$. A line segment is then drawn parallel to the x-axis from A over to the y-axis, which the segment meets at point B. Points O, A, B are thus the vertices of a right triangle. Deduce the value of m that maximizes the area R of triangle OAB, and prove that this really is the maximum area.

2. Let a be a positive real number. The Lemniscate of Bernoulli is defined by

$$(x^2 + y^2)^2 = a^2(x^2 - y^2).$$

Find the area bounded by the Lemniscate of Bernoulli.

3. The sequence of Catalan numbers, $\{C_n\}_{n=1}^{\infty}$, is defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Does there exist a member of the sequence that is not a natural number? Find one, or prove that there is none.

4. Suppose a belt is stretched tightly over two circular pulleys with radii r_1 and r_2, whose centers are d units apart with $d > r_1 + r_2$. If $r_1 > r_2$, find a formula for the total length of the belt in terms of r_1, r_2, and d.

5. Evaluate

$$\sum_{k=1}^{\infty} \frac{1}{\binom{k+n}{k}}$$

for $n \geq 2$. What is this series when $n = 1$?

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Session II

1.

(a) Let $f(x) = x^3 + x$. Let $g(x)$ be the inverse function of $f(x)$. Find $g'(10)$.

(b) For $x > 0$, define $h(x) = 1/f(x)$. Prove that the function $f(x) + h(x)$ has its absolute minimum when $x = g(1)$.

2. Consider the lattice $L = \{(x, y) \ : \ x, y \in \mathbb{Z}\}$. Color the lattice using an arbitrary coloring scheme with 2008 available colors. Prove or disprove: In every coloring scheme, there must be a rectangle whose four vertices all lie in L and are colored with the same color.

3. Find $b > 1$ such that the graphs of $y = \log_b(x)$ and $y = b^x$ intersect in exactly one point, i.e., are tangent to one another.

4. Suppose that

$$\frac{2x+3}{x^2 - 2x + 2}$$

has the Taylor series

$$\sum_{k=0}^{\infty} a_k x^k.$$

Find the sum of the odd numbered coefficients, i.e., find

$$\sum_{k=0}^{\infty} a_{2k+1} = a_1 + a_3 + a_5 + \cdots .$$

5. For each integer n, let $a_n = 8n^2 + 3n + 10$ and $b_n = 3n^2 + n + 3$. Since $a_1 = 21$ and $b_1 = 7$, we can write $\gcd(a_1, b_1) = 7$, where gcd denotes the greatest common divisor. Find $\max_{n \in \mathbb{Z}} \gcd(a_n, b_n)$.