## 2007 Missouri Collegiate Mathematics Competition

### Session I

1. Seemingly unimportant changes in the terms of a series can have a dramatic effect on the nature of the sum. Thus, the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

sums to a transcendental number, but if the $n^2$ is replaced by $n^2 + 3n - 4$, the new series sums to a rational number. Find it.

2. A bitwin chain of length one consists of two pairs of twin primes with the property that they are related by being of the form:

$$(n-1, n+1) \text{ and } (2n-1, 2n+1).$$

Prove or disprove that $30 \mid n$ for $n \geq 7$.

3. Let $f$ be the complex function

$$f(z) = e^{z^2}.$$

Find the set of all points $z$ in the complex plane for which $f(z)$ is a real number. Give a geometric description of this set.

4. A triangle is Pythagorean if it is a right triangle and the lengths of all of its sides are integers. Suppose that $\triangle ABC$ is Pythagorean; for concreteness assume that the lengths of the three sides satisfy $c > a > b$. The median and the altitude are now drawn from $C$ to the hypotenuse, where they meet the latter at $P$, $Q$, respectively. Determine simple conditions upon $a$, $b$, $c$ so that $\triangle CQP$ will also be Pythagorean.

5. Associated with the Fibonacci numbers $\{F_n\}_{n=1}^{\infty}$, are the Fibonacci polynomials, $\{U_n(x)\}_{n=1}^{\infty}$ ,

$$U_n(x) = \begin{cases} 1, & n = 1; \\ x, & n = 2; \\ xU_{n-1}(x) + U_{n-2}(x), & n > 2. \end{cases}$$

(a) Find a function $F(x, y)$ such that (formally)

$$F(x, y) = \sum_{n=1}^{\infty} U_n(x) y^n.$$

Such a function is called a generating function for the $U_n(x)$'s.

(b) Establish that for each $n$, $U_n(1) = F_n$.

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## Session II

1. Suppose that

$$\frac{2x+3}{x^2-2x+2}$$

has the Taylor series

$$\sum_{k=0}^{\infty} a_k x^k.$$

Find

$$\sum_{k=0}^{\infty} a_k.$$

2. Find the point on a given line such that the sum of its distances from two fixed points is a minimum. Assume the two fixed points and the given line are in the same plane.

3. Assume the point $(h, k)$ lies "outside" the circle $x^2 + y^2 = r^2$, $0 < r < \sqrt{h^2 + k^2}$. Find an expression for the $x$-intercepts of the tangent lines to the circle from the point $(h, k)$ in terms of $h$, $k$, and $r$.

4. Let

$$S = \{5a + 503b \ : \ a \text{ and } b \text{ are nonnegative integers}\}.$$

What is the largest integer which does NOT belong to $S$?

5. Does there exist a real valued continuous function $f$ with domain the set of all real numbers (usual topology on the domain and range) such that if $x$ is rational then $f(x)$ is irrational and if $x$ is irrational then $f(x)$ is rational? Prove your answer.