2006 Missouri Collegiate Mathematics Competition

Session I

1. Find

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+4}+\frac{n}{n^2+9}+\cdots+\frac{1}{2n}\right).$$

2. You have a supply of unit squares, 1×2 rectangles and 1×3 rectangles. With these you wish to "tile" a rectangular strip of dimensions $1\times n$ (n is a positive integer). For example, if W_n is the number of ways a $1\times n$ strip can be tiled, then $W_3=4$ since

$$3=2+1=1+2=1+1+1.$$

Determine how many tiling patterns W_n exist when $n=16$ and prove your answer.

3. Let $\{a_n\}$ be the sequence defined recursively by

$$a_0=1,\ a_1=0,\ \text{and } a_n=\frac{1}{n}a_{n-2}\ \text{ for }\ n\geq 2.$$

If the function f is defined by

$$f(x)=\sum_{n=0}^{\infty}a_nx^n,$$

find the exact value of $f(2)$.

4. Find all positive integers c such that $n(n+c)$ is never a perfect square for any positive integer n.

5. Let $f(t)$ and $f'(t)$ be differentiable on $[a,x]$ and for each x suppose there is a number c_x such that $a<c_x<x$ and

$$\int_a^x f(t)\,dt=f(c_x)(x-a).$$

Assume that $f'(a)\neq 0$. Then prove that

$$\lim_{x\to a}\frac{c_x-a}{x-a}=\frac{1}{2}.$$

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Session II

1. The array below is called a magic square because the sum of the three numbers along any row, any column, or the two diagonals, is the same (namely, 15).

8	1	6
3	5	7
4	9	2

(a) Construct a 3×3 multi-magic square, that is, a 3×3 array of 9 distinct integers such that the PRODUCT of the three numbers along any row, any column, or the two diagonals, is the same.

(b) Show that no multi-magic square can be constructed with nine *consecutive* integers.

2. Evaluate the limit:

$$\lim_{n\to\infty} \frac{1}{n^4} \sum_{k=3}^{n} \binom{k}{3}.$$

3. Define a sequence of positive integers $\{x_n \mid n = 1, 2, 3, \ldots\}$ to be *Dence* if every positive integer can be expressed as a sum of distinct members of the sequence. Now consider the sequence in which $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, $x_4 = 7$, $x_5 = 15$, and $x_{k+1} = 2x_k - 7$ for all $k \geq 5$. Prove that this sequence is Dence.

4. For a convex polygon with n sides, let T be a point in the interior of the polygon. Triangulate the polygon by drawing line segments from T to each vertex. Denote the distance from T to side s_i of the polygon by r_i and the area of the corresponding triangle by A_i. Let A be the total area of the polygon. Show that the number r, defined by

$$\frac{1}{r} = \sum_{i=1}^{n} \left(\frac{A_i}{A}\right)\left(\frac{1}{r_i}\right),$$

is independent of the position (inside the polygon) of T.

5. Let

$$\{a_n\}_{n=1}^{\infty} \quad \text{and} \quad \{\Delta a_n\}_{n=1}^{\infty} = \{a_n - a_{n+1}\}_{n=1}^{\infty}$$

be two decreasing sequences of positive numbers that converge to 0. Prove that the magnitude of the error, $|R_n|$, made in approximating the sum of the series

$$\sum_{k=1}^{\infty}(-1)^{k-1}a_k$$

by its n partial sum is bounded as follows:

$$\frac{a_{n+1}}{2} < |R_n| < \frac{a_n}{2}.$$