2005 Missouri Collegiate Mathematics Competition

Session I

1. Find the point in the first quadrant on the graph of $y = 7 - x^2$ such that the distance between the x- and y- intercepts of the tangent line at the point is minimum.

2. Let $M(n,k)$ be the number of mappings from a set X of n distinct objects onto a set Y of k distinct objects, and let $P(n,k)$ denote the number of partitions of a set of n distinct objects into k nonempty subsets. Determine the relationship between $M(n,k)$ and $P(n,k)$ and use it to show that $M(n,k)$ is a multiple of 24 whenever $k > 3$.

3. Suppose that f is a polynomial of positive degree n with integer coefficients. Prove that there are infinitely many integers x for which $f(x)$ is composite. (Here, composite means those integers, positive or negative, whose absolute value is not 1 or a prime; thus, -4 and 6 are composite, while 1 and -2 are not.)

4. Determine the value of the integral

$$I(\theta) = \int_{-1}^{1} \frac{\sin\theta \, dx}{1 - 2x\cos\theta + x^2},$$

and locate those points $0 \leq \theta \leq 2\pi$, where $I(\theta)$ is discontinuous.

5. Prove that in the MacLaurin series for $\tan\theta$, $-\pi/2 < \theta < \pi/2$, every coefficient is non-negative.

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Session II

1. If $a < b < c$, $f'(x)$ is strictly increasing on (a, c), and $f(x)$ is continuous on $[a, c]$, then show that

$$(b - a)f(c) + (c - b)f(a) > (c - a)f(b).$$

2. Find all integer solutions (x, y) to the equation $xy = 5x + 11y$.

3. Define a circle of radius r and center P on a sphere to be the locus of points on the surface of the sphere that are a distance r from P, where distance is the usual Euclidean distance in $\mathbb{R}^3$. When r is less than the diameter of the sphere, this circle divides the sphere into two spherical segments, or "caps". Show that the area of the cap containing the center point P is πr^2.

4. Let $p > 2$ be a prime. Prove or disprove that all prime divisors of $2^p - 1$ have the form $2kp + 1$.

5. Suppose that $f\colon [0, \infty) \to [0, \infty)$ is a differentiable function with the property that the area under the curve $y = f(x)$ from $x = a$ to $x = b$ is equal to the arclength of the curve $y = f(x)$ from $x = a$ to $x = b$. Given that $f(0) = 5/4$, and that $f(x)$ has a minimum value on the interval $(0, \infty)$, find that minimum value.