2002 Missouri MAA Collegiate Mathematics Competition

Session I

1. Let $P \neq (0, 0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at P will intersect the parabola at another point, say Q. Find the coordinates of P so that the average of the y-coordinates of P and Q is a minimum.

2. A tetrahedron is called *isosceles* if the members of each pair of opposite edges are equal. This means, for tetrahedron $ABCD$, that $AB = CD$, $BC = AD$, and $AC = BD$.

(a) Prove that all four faces of an isosceles tetrahedron are congruent.
(b) Prove that if all four faces of a tetrahedron have the same perimeter, then the tetrahedron is isosceles.
(c) Prove that a tetrahedron is isosceles if and only if the sum of the face angles at each vertex is 180°.

3. Let $\{x_n\}$ be the following sequence involving alternating square roots of 5 and 13:

$$x_1 = \sqrt{5}, \quad x_2 = \sqrt{5 + \sqrt{13}}, \quad x_3 = \sqrt{5 + \sqrt{13 + \sqrt{5}}}, \quad x_4 = \sqrt{5 + \sqrt{13 + \sqrt{5 + \sqrt{13}}}},$$

and so on. Prove that $\lim_{n\to\infty} x_n$ exists and determine its value.

4. Does the set $X = \{1, 2, \ldots, 3000\}$ contain a subset A of 2000 integers in which no member of A is twice another member of A?

5. Two right circular cylinders of radius r intersect at right angles to form a solid. This solid has four curved faces. Imagine one of these faces "rolled out flat". Find equations of the boundary curves of this flattened face and also find its area.

2002 Missouri MAA Collegiate Mathematics Competition

Session II

1. Seven golf balls, labeled 1 through 7, are correctly placed in corresponding boxes (one to a box), also labeled 1 through 7. The balls are now removed and then randomly returned to the boxes, one ball to a box. What is the probability that no ball will find its correct box?

2.

(a) Prove that, for any positive integer n,

$$\sin n\theta = \binom{n}{1}\sin\theta\cos^{n-1}\theta - \binom{n}{3}\sin^3\theta\cos^{n-3}\theta + \binom{n}{5}\sin^5\theta\cos^{n-5}\theta - \cdots$$

and

$$\cos n\theta = \cos^n\theta - \binom{n}{2}\sin^2\theta\cos^{n-2}\theta + \binom{n}{4}\sin^4\theta\cos^{n-4}\theta - \cdots .$$

(b) Prove that, for all x in the interval $[-1, 1]$ and for any positive integer n, the function

$$T_n(x) = \cos(n\cos^{-1}x)$$

is a polynomial in x of degree n and leading coefficient 2^{n-1}.

3. Suppose three equal circles, each of radius r, pass through a common point O and have three other pairwise intersections at P_1, P_2, and P_3. Prove that the circle containing P_1, P_2, and P_3 also has radius r.

4. $ABCDE$ is a regular pentagon of side s, and P is any point in the interior of $ABCDE$. Line segments are drawn from P perpendicular to each of the five sides. Denote the sum of the lengths of these five perpendiculars by S. Prove that S is independent of the location of P, and find S in terms of s.

5. Let $p(n)$ denote the product of the (decimal) digits of the positive integer n. Consider the sequences, beginning at any arbitrary positive integer, in which succeeding terms are obtained by adding to the previous term the product of its digits:

$$n_0 = n, \quad \text{and for} \quad r \geq 0, \quad n_{r+1} = n_r + p(n_r).$$

Is there an initial integer n for which the sequence continues to increase indefinitely?