2000 Missouri MAA Collegiate Mathematics Competition

Session I

1. Let $P \neq (0,0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at P will intersect the parabola at another point, say Q. Find the coordinates of P so that the distance between the x-coordinates of P and Q is a minimum.

2. If $xyz = (1-x)(1-y)(1-z)$ where $0 \leq x, y, z \leq 1$, show that

$$x(1-z) + y(1-x) + z(1-y) \geq \frac{3}{4}.$$

3. Let $n \geq 3$ points be given in the plane. Prove that three of them form an angle which is at most π/n.

4. Justify as far as you can, the equality

$$\int_0^1 x^x\,dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \cdots.$$

5. Show that a polynomial in x with real coefficients which takes rational values for rational arguments and (real) irrational values for (real) irrational arguments must be linear.

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Session II

1. Two points are chosen at random (with a uniform distribution) from the unit interval $[0, 1]$. What is the probability that the points will be within a distance of ϵ of each other?

2. Write Pascal's Triangle as an infinite array as follows:

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 3 & 6 & 10 & 15 & \cdots \\ 1 & 4 & 10 & 20 & 35 & \cdots \\ 1 & 5 & 15 & 35 & 70 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

where the first row and first column consist entirely of ones and every other entry is formed by taking the sum of the entry to the left and the entry above. For each positive integer n, let D_n denote the n by n matrix formed by the first n rows and first n columns of the array. What is the determinant of D_n? Prove your answer.

3. Given an $n \times n$ checkerboard with the four corners removed, characterize for which n this deleted board can be covered with 3×1 rectangles.

4. For $n \geq 2$, let $x_1, \ldots, x_n$ be non-zero real numbers whose sum is zero. Show that there are i, j with $1 \leq i < j \leq n$ such that

$$1/2 \leq |x_i/x_j| \leq 2.$$

5. This problem concerns sequences $x_1 x_2 \cdots x_n$ in which each x_i is either a, b, or c. Determine the number of those sequences which have length n, begin and end with the letter a, and in which adjacent terms are always different letters.