1999 Missouri MAA Collegiate Mathematics Competition

Session I

1. Let $P \neq (0,0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at P will intersect the parabola at another point, say Q. Find the coordinates of P so that the length of the arc of the parabola between P and Q is a minimum.

2. Give a precise characterization of those points in the plane which do not lie on a tangent line to the curve $y = x^4 - 6x^2$.

3. On a 5×5 square matrix place 13 black counters and 12 white counters in alternating checkerboard fashion. Remove the black counter in the center square. Player A controls the white counters and B the black. They take turns moving one of their counters orthogonally to the vacant square until a player loses by being unable to move. Which player has a winning strategy? What is the strategy?

4. If $x_0 = 5$ and $x_{n+1} = x_n + 1/x_n$, prove that for all $n \geq 1$

$$2n < x_n^2 - 25 < 47n/23.$$

5. For an $n \times n$ matrix X, if $\det(\lambda I - X) = 0$ then we say λ is an eigenvalue of X. Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix. Prove that AB and BA have the same non-zero eigenvalues.

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Session II

1. Let SC be the semicircle with $y \geq 0$ centered at $(1,0)$ with radius 1. Let C_a be the circle with radius $a > 0$ and center $(0,0)$ and denote the point $(0,a)$ by P. Consider the line through P and the intersection of SC and C_a. What is the limiting position of the x-intercept of this line as $a \to 0$?

2. Find the limit

$$\lim_{N\to\infty}\left(1 - 2\sum_{n=1}^{N}\frac{1}{16n^2-1}\right).$$

3. For n positive real numbers with minimum m and maximum M, let A and G denote their arithmetic and geometric means. Prove that

$$A - G \geq n^{-1}(\sqrt{M} - \sqrt{m})^2.$$

4. Find all possible continuous and differentiable curves C which have the following properties. The curve C lies in the first quadrant and contains the point $(0,0)$. Whenever P is on C the interior of the rectangle R bounded by the coordinate axes and horizontal and vertical lines through P is separated into two parts by C. When the part adjacent to the x-axis is rotated about the x-axis and the part adjacent to the y-axis is rotated about the y-axis, two solids of equal volume are generated.

5. Let A_n denote the $n \times n$ matrix whose (i,j) entry is $\mathrm{GCD}(i,j)$. Compute $\det(A_n)$.