1998 Missouri MAA Collegiate Mathematics Competition

Session I

1. Let $P \neq (0,0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at P will intersect the parabola at another point, say Q. Find the coordinates of P so that the length of segment PQ is a minimum.

2. Let $0 < a_1 < a_2 < \cdots < a_n$ and let $e_i = \pm 1$. Prove that $\sum_{i=1}^{n} e_i a_i$ assumes at least $\binom{n+1}{2}$ distinct values as the e_i's range over the 2^n possible combinations of signs.

3. If m and n are positive integers and $a < b$, find a formula for

$$\int_a^b \frac{(b-x)^m}{m!} \frac{(x-a)^n}{n!} dx$$

and use your formula to evaluate

$$\int_0^1 (1-x^2)^n dx.$$

4. Describe how to fold a rectangular sheet of paper so that the lower right corner touches the left edge and the length of the crease is a minimum. Discuss how the dimensions of the rectangle affect the result.

5. Let n be a positive integer. Prove that there exists a number divisible by 5^n that does not contain a single zero in its decimal notation.

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Session II

1. Let I be the $n \times n$ identity matrix. Prove that $AB - BA \neq I$ for any $n \times n$ matrices A and B.

2. Let $a_1, \ldots, a_n$ be positive real numbers and let s denote their sum. Show that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \leq 1 + \frac{s}{1!} + \frac{s^2}{2!} + \cdots + \frac{s^n}{n!}.$$

3. Sum the series

$$\sum_{i=1}^{\infty} \frac{36i^2 + 1}{(36i^2 - 1)^2}.$$

4. Circle O has a diameter of 3; let AOD be a diameter. A second circle, of radius 1, is inscribed in circle O so that its center lies along AOD and such that this circle is tangent to circle O at point D. A third circle, of radius 1/2, is next inscribed in circle O so that its center also lies along AOD and such that it is tangent to circle O at point A. Determine the radius of a fourth circle that could be constructed inside circle O and which would be simultaneously tangent to all three circles.

5. A sequence of polynomials $\{P_i(x)\}_{i=0}^{\infty}$ is defined by the generating function

$$\frac{2e^{tx}}{e^t + 1} = \sum_{i=0}^{\infty} P_i(x) \frac{t^i}{i!}.$$

Show that 1 is a zero of $P_i(x)$ for all even $i > 0$, and 1/2 is a zero of $P_i(x)$ for all odd $i > 0$.