1997 Missouri MAA Collegiate Mathematics Competition

Session I

1. Let $P \neq (0,0)$ be a point on the parabola $y = x^2$. The normal line to the parabola at P will intersect the parabola at another point, say Q. Find the coordinates of P so that the y-coordinate of Q is a minimum.

2. Prove that from any row of n integers one may always select a block of adjacent integers whose sum is divisible by n.

3. Find conditions on the parameters a, b, c, and d so that

$$f(x,y) = a\sin(x+y) + b\cos(x+y) + c\sin(x-y) + d\cos(x-y)$$

can be written as $f(x,y) = g(x)h(y)$.

4. A point P is in the interior of a circle of radius r. Place the vertex of a right angle at P and denote by A and B the points where the sides of the right angle intersect the circle. Let Q be the point which completes the rectangle $PAQB$. What is the locus of Q?

5. Let $\{L_n\}_{n=0}^{\infty}$ be the sequence of Lucas numbers: $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Let $DR(N)$ denote the digital root of a positive integer N, defined as the sum of the digits of N, composed enough times until a value between 1 and 9 is obtained. For example, $DR(667) = DR(19) = DR(10) = 1$. Show that there is a smallest positive integer k such that $DR(L_{n+k}) = DR(L_n)$ for all integers $n \geq 0$.

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Session II

1. Find positive integers n and $a_1, a_2, \ldots, a_n$ such that

$$a_1 + a_2 + \cdots + a_n = 1997$$

and the product $a_1 a_2 \cdots a_n$ is as large as possible.

2. Let a, b, c, d be positive numbers with $abcd = 1$. Prove that

$$a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd \geq 10.$$

3. A wooden cube of edge 3 is formed by gluing together 27 small cubes of edge 1. A termite, beginning with any one of the outer small cubes, begins to eat its way through the large cube, always moving perpendicular to a face (i.e., no diagonal movements are allowed - don't ask why, who knows the mind of a termite?) Is it possible for the termite to follow a path entirely within the large cube (emerging and crawling on the outside is also not allowed) which passes through each small cube exactly once and ends in the center cube? Generalize the problem to the case where the large cube has edge n, an odd integer.

4. Define a family of curves by

$$S_n = \{(x, y) : y = \frac{1}{n}\sin(n^2 x),\ 0 \leq x \leq \pi\},$$

where n is a positive integer. What is the limit of the length of S_n as $n \to \infty$?

5. Consider the infinite sequences $\{x_n\}$ of positive real numbers with the following properties:

$$x_0 = 1, \quad \text{and for all } i \geq 0, \quad x_{i+1} \leq x_i.$$

(a) Prove that for every such sequence, there is an $n \geq 1$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

(b) Find such a sequence for which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} < 4 \text{ for all } n.$$