1. If a polygon has both an inscribed circle and a circumscribed circle, then define the *halo* of that polygon to be the region inside the circumcircle but outside the incircle. In particular, all regular polygons and all triangles have halos.

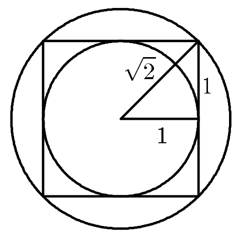
   (a) What is the area of the halo of a square with side length 2?

   (b) What is the area of the halo of a 3-4-5 right triangle?

   (c) What is the area of the halo of a regular 2016-gon with side length 2?

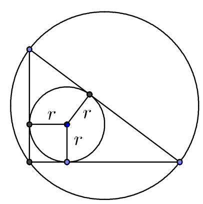
Solutions to Problem #1.

Part (a)



Area of the halo is $\pi\left(\sqrt{2}\right)^2 - \pi(1)^2 = \pi$

Part (b)

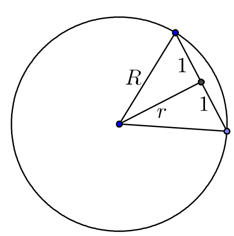


The hypotenuse is a diameter of the circumscribed circle. So its radius is $\frac{5}{2}$. To find the radius of the inscribed circle, compute the area of the triangle in two ways:

$$\begin{aligned} \text{Area of triangle is } \frac{1}{2}(4\cdot 3) &= \frac{1}{2}r(3+4+5) \\ 6 &= 6r \\ 1 &= r \end{aligned}$$

Hence the area of the halo is $\pi\left(\frac{5}{2}\right)^2 - \pi(1)^2 = \frac{21}{4}\pi$

Part (c).



Consider a regular polygon with n sides each of length 2.
Let R denote the radius of the circumscribed circle and let r denote the radius of the inscribed circle. By the Pythagorean theorem, $r^2 + 1 = R^2$. Hence the area of the halo is

$$\pi R^2 - \pi r^2 = \pi$$

Note that the area of the halo is independent of n. Hence, the area of the halo of a regular 2016-gon with side length 2 is π.

2. Let $s_1, s_2, s_3, s_4, \ldots$ be a sequence (infinite list) of 1's and 0's. For example $1, 0, 1, 0, 1, 0, \ldots$, that is, $s_n = 1$ if n is odd and $s_n = 0$ if n is even, is such a sequence. Prove that it is possible to delete infinitely many terms in $s_1, s_2, s_3, s_4, \ldots$ so that the resulting sequence is the original sequence. For the given example, one can delete $s_3, s_4, s_7, s_8, s_{11}, s_{12}, \ldots$.

Solutions to Problem #2. If the sequence has finitely many 1's, then there exists an m such that $a_i = 0$ for all $i \geq m$; thus we can delete even i's when $i \geq m$. Similarly if the sequence has finitely many 0's. Thus we may assume that the sequence contains infinitely many 0's and infinitely many 1's. We prove the claim by construction. It is easier to describe the algorithm by first creating an identical sequence q_n, that is, $q_i = s_i$ for all i. We keep q_1. The next step is to delete $q_2, \ldots, q_j$ where $j > 2$ such that $q_{j+1} = b_2$. This is possible because the sequence has infinitely many 0's and 1's. In general, after deleting terms from q_n's so that the initial k terms of the resulting sequence match the ones in s_n's, delete at least one term after the kth term in the revised q_n's until the next term is the same as b_{k+1}. This is possible because the sequence has infinitely many 0's and 1's and up to this point we have only deleted finitely many terms. Since we delete at least one term in each step, infinitely many terms will be deleted.

3. This problem is about pairs of consecutive whole numbers satisfying the property that one of the numbers is a perfect square and the other one is the double of a perfect square.

(a) The smallest such pairs are $(0,1)$ and $(8,9)$, Indeed $0 = 2 \times 0^2$ and $1 = 1^2$; $8 = 2 \times 2^2$ and $9 = 3^2$. Show that there are infinitely many pairs of the form $(2a^2, b^2)$ where the smaller number is the double of a perfect square satisfying the given property.

(b) Find a pair of integers satisfying the property that is not in the form given in the first part, that is, find a pair of integers such that the smaller one is a perfect square and the larger one is the double of a perfect square.

Solution to Problem #3.

(a) We say that the pair $(2a^2, b^2)$ is *associated to* the pair (a, b). From the relation $2a^2 + 1 = b^2$ it follows that $2a^2 = b^2 - 1 = (b-1)(b+1)$. This implies first that b has to be an odd number, and this further implies that a has to be an even number, because $(b-1)(b+1)$ is a multiple of 4. Let $a = 2s$ and $b = 2t + 1$. Introducing these in the relation above, we get: $8s^2 = 4t(t+1)$, or $2s^2 = t(t+1)$. The product of the two consecutive numbers suggest that we could use for t the value $2\alpha^2$ from one of the already known pairs associated to (α, β). Substituting in the previous equation, we get $2s^2 = 2\alpha^2\beta^2$, or $s = \alpha\beta$. The outcome of these observations is that from a pair associated to (α, β) we obtain another such pair, this time associated to $(2\alpha\beta, 4\alpha^2 + 1)$. For example, if we start with the pair $(8, 9)$ associated to $(2, 3)$, we obtain the new pair $(288, 289)$, associated to $(12, 17)$. Finally, we note that when passing from (α, β) to $(2\alpha\beta, 4\alpha^2 + 1)$ the first component of these pairs is increasing, so we do obtain inductively an infinite sequence of pairs with the desired property.

Note. There is no implication that the method presented above finds *all* the pairs $(2a^2, b^2)$.

(b) $(49, 50)$ is such an example. The simplest way to find this is probably by writing a few terms of the two sequences $\{n^2\}_n$ and $\{2n^2\}_n$:

$$\{n^2\}_{n\geq 0}: \quad 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, \ldots$$
$$\{2n^2\}_{n\geq 0}: \quad 0, 2, 8, 18, 32, 50, 72, 98, 128, 162, \ldots$$

One can now easily observe the two examples mentioned in the statement and the new one $(49, 50)$.

4. It is a fact that every set of 2016 consecutive integers can be partitioned in two sets with the following four properties:
(i) The sets have the same number of elements.
(ii) The sums of the elements of the sets are equal.
(iii) The sums of the squares of the elements of the sets are equal.
(iv) The sums of the cubes of the elements of the sets are equal.

Let $S = \{n+1, n+2, \ldots, n+k\}$ be a set of k consecutive integers.

(a) Determine the smallest value of k such that property (i) holds for S.

(b) Determine the smallest value of k such that properties (i) and (ii) hold for S.

(c) Show that properties (i), (ii) and (iii) hold for S when $k = 8$.

(d) Show that properties (i), (ii), (iii) and (iv) hold for S when $k = 16$.

Solutions to Problem #4.

Part (a) Property (i) holds for the partition $\{n+1\}$ and $\{n+2\}$, so $k = 2$ suffices. Furthermore, if $k = 1$, then property (i) cannot hold for S. So $k = 2$ is minimal. Note that the sum of the elements for the second set is 1 larger than in the first set, which lends itself nicely to...

Part (b) Properties (i) and (ii) hold for the partition $\{n+1, n+4\}$ and $\{n+2, n+3\}$, so $k = 4$ will suffice. Furthermore, if $k = 1$ or 3, then property (i) cannot hold, and if $k = 2$, then property (ii) cannot hold. So $k = 4$ is minimal. Note that the sum of the squares of the elements of the first set is exactly 4 bigger than in the second set, which lends itself nicely to...

Part (c) Properties (i), (ii), and (iii) hold for the partition $\{n+1, n+4, n+6, n+7\}$ and $\{n+2, n+3, n+5, n+8\}$, so $k = 8$ suffices. Note that the sum of the cubes of the elements of the first set is exactly 48 smaller than the second set, which lends itself nicely to...

Part (d) Properties (i), (ii), (iii), and (iv) hold for the partition $\{n+1, n+4, n+6, n+7, n+10, n+11, n+13, n+16\}$ and $\{n+2, n+3, n+5, n+8, n+9, n+12, n+14, n+15\}$, so $k = 16$ suffices. It is noteworthy that this process extends quite nicely into higher powers of 2. In fact, because 2016 is divisible by 32, we could extend this all the way up to "sums of fourth powers of elements".

5. Consider four real numbers x, y, a, and b, satisfying $x + y = a + b$ and $x^2 + y^2 = a^2 + b^2$. Prove that $x^n + y^n = a^n + b^n$, for all $n \in \mathbb{N}$.

Solutions to Problem #5.

Method 1. Fix a and b and try to understand what values x and y could possibly get. Are there infinitely many such numbers? Well, the disappointing answer is "No." We will show that they can only take the values a and b, that is, either $x = a$ and $y = b$, or $x = b$ and $y = a$. In either case, it follows that the relation that needs to be proved is trivially satisfied. Square $x+y = a+b$ and use the second given relation to obtain $xy = ab$. This means that x and y are the solutions of the same quadratic equation that a and b satisfy, $z^2 - (a+b)z + ab = 0$. Consequently x and y have to equal the only roots possible, namely a and b.

Note. The above proof shows that the problem is actually less interesting than the statement may indicate! The next proof shows the identity without questioning what numbers x, y, a, b could possibly satisfy the hypothesis of the problem.

Method 2. Proof by generalized induction. Let $P(n)$ be the statement that we have to prove, $x^n + y^n = a^n + b^n$. The first given relation shows that $P(1)$ holds. As above, we also have $xy = ab$. Now assume $P(k)$ holds for all $1 \leq k \leq n$. Raise $x + y = a + b$ to power $n + 1$ and use the symmetry of the binomial coefficients to write:

$$x^{n+1} + y^{n+1} + \binom{n+1}{1} xy(x^{n-1} + y^{n-1}) + \binom{n+1}{2} x^2y^2(x^{n-3} + y^{n-3}) + \cdots =$$
$$= a^{n+1} + b^{n+1} + \binom{n+1}{1} ab(a^{n-1} + b^{n-1}) + \binom{n+1}{2} a^2b^2(a^{n-3} + b^{n-3}) + \ldots$$

By the induction hypothesis, all the terms, except the first two, cancel. So $P(n + 1)$ also holds.