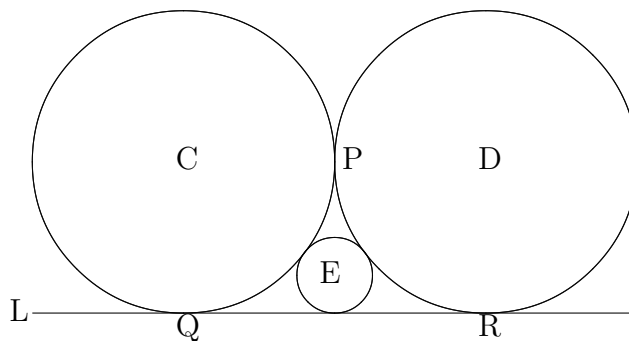


1. A permutation on  $\{1, 2, \dots, n\}$  is an ordered arrangement of the numbers. For example, 32154 is a permutation of  $\{1, 2, 3, 4, 5\}$ . Does there exist a permutation  $a_1 a_2 \dots a_n$  of  $\{1, 2, \dots, n\}$  such that  $i + a_i$  is a perfect square for  $1 \leq i \leq n$  when a)  $n = 6$ , b)  $n = 13$ , c)  $n = 86$ ? Justify your answers.

*Solution:*

- a) No. The conditions force  $a_1 = 3$  and  $a_6 = 3$ .
- b) Yes, take  $(a_1, \dots, a_{13}) = (8, 2, 13, 12, 11, 10, 9, 1, 7, 6, 5, 4, 3)$ .
- c) Yes. Note that  $10^2 = 100$ , and take  $a_i = 100 - i$  for  $14 \leq i \leq 100$ . For the remaining  $i$ , use (b) and take  $(a_1, \dots, a_{13}) = (8, 2, 13, 12, 11, 10, 9, 1, 7, 6, 5, 4, 3)$

2. Circle  $C$  and circle  $D$  are tangent at point  $P$ . Line  $L$  is tangent to  $C$  at point  $Q$  and to  $D$  at point  $R$  where  $Q$  and  $R$  are distinct from  $P$ . Circle  $E$  is tangent to  $C$ ,  $D$ , and  $L$ , and lies inside triangle  $PQR$ .  $C$  and  $D$  both have radius 8. Find the radius of  $E$ , and justify your answer.



*Solution:* Let  $a$  be the radius of circle  $E$ , and let points  $C$ ,  $D$ , and  $E$  be the centers of the respective circles with those labels. Then triangles  $CPE$  and  $DPE$  are congruent by  $SSS$ , so angle  $CPE$  is a right angle.

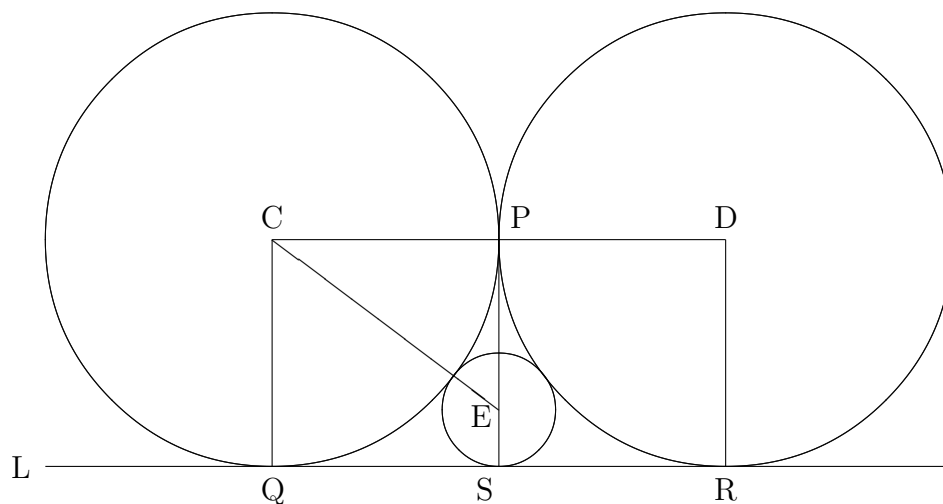
Consider triangle  $CPE$ . By the Pythagorean Theorem,

$$CE^2 = CP^2 + PE^2.$$

We have  $CP = 8$ ,  $CE = 8 + a$ , and  $PE = 8 - a$ .

[To see the second equality, note that segment  $CE$  consists of collinear radii of circles  $C$  and  $E$ . For the last equality, consider the midpoint  $S$  of segment  $QR$ . Segment  $PE$  part of segment  $PS$ .  $CPSQ$  is a square of side 8, and  $ES$  has length  $a$ .]

Thus  $(8 + a)^2 = 8^2 + (8 - a)^2$ . It follows that  $a = 2$ .



3. (a) Prove that  $\sin 3x = 4 \cos^2 x \sin x - \sin x$  for all real  $x$ .  
 (b) Prove that

$$(4 \cos^2 9^\circ - 1)(4 \cos^2 27^\circ - 1)(4 \cos^2 81^\circ - 1)(4 \cos^2 243^\circ - 1)$$

is an integer.

*Solution:*

- (a) By the addition formulas for  $\sin$  and  $\cos$ ,  $\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1$ ,  $\sin 2x = 2 \sin x \cos x$ , and

$$\begin{aligned} \sin 3x &= \sin 2x \cos x + \cos 2x \sin x \\ &= 2 \sin x \cos^2 x + (2 \cos^2 x - 1) \sin x \\ &= 4 \cos^2 x \sin x - \sin x. \end{aligned}$$

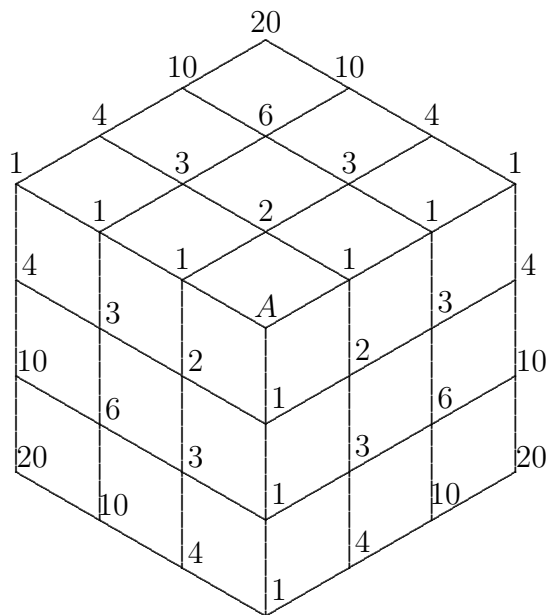
- (b) By part (a),  $4 \cos^2 x - 1 = \sin 3x / \sin x$  whenever  $\sin x \neq 0$ . We apply this to see that the given expression is

$$\frac{\sin 27^\circ}{\sin 9^\circ} \frac{\sin 81^\circ}{\sin 27^\circ} \frac{\sin 243^\circ}{\sin 81^\circ} \frac{\sin 729^\circ}{\sin 243^\circ} = \frac{\sin 729^\circ}{\sin 9^\circ} = 1,$$

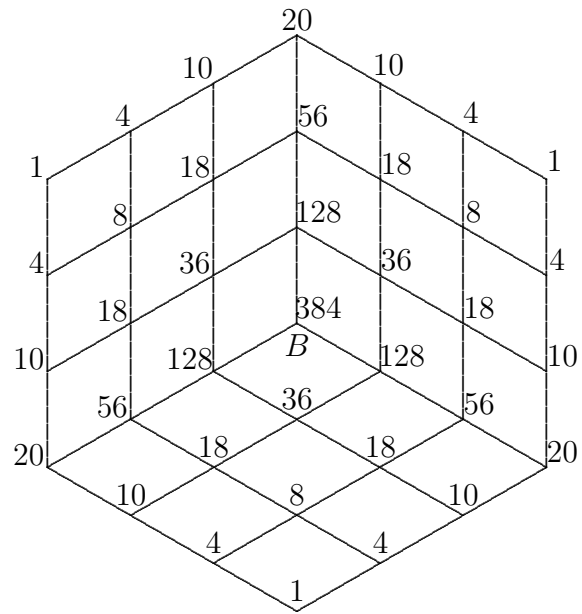
which is an integer.

4. Consider a  $3 \times 3 \times 3$  stack of small cubes making up a large cube (as with the small cubes in a Rubik's cube). An ant crawls on the surface of the large cube to go from one corner of the large cube to the opposite corner. The ant walks only along the edges of the small cubes and covers exactly nine of these edges. How many different paths can the ant take to reach its goal?

*Solution:* Start at corner  $A$  and determine the number of paths to vertices as successively farther distances from  $A$ . For the three faces of the large cube that contain  $A$ , we have:

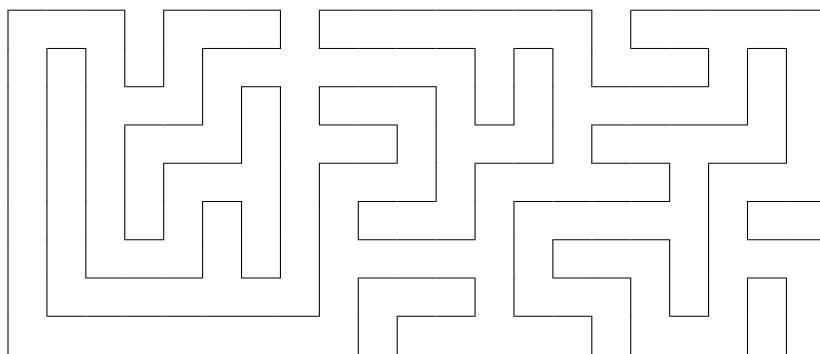


Then continue on the three faces that contain the ending point  $B$  working from the outside successively closer to  $B$ . We have:



Thus there are 384 different paths the ant can take to reach its goal.

5. Let  $m$  and  $n$  be positive integers, and consider the rectangular array of points  $(i, j)$  with  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ . For what pairs  $m, n$  of positive integers does there exist a polygon for which the  $mn$  points  $(i, j)$  are its vertices, such that each edge is either horizontal or vertical? The figure below depicts such a polygon with  $m = 10$ ,  $n = 22$ . Thus 10, 22 is one such pair.



*Solution.* The polygon is made of  $mn$  segments of unit length. When moving from  $(i, j)$  to  $(i \pm 1, j)$  or to  $(i, j \pm 1)$ , the parity of the sum of the coordinates is reversed. Since the parity is the same at the end as it was initially,  $mn$  must be even. Also, we must have  $m > 1$  and  $n > 1$ . To see that these conditions are sufficient, consider the following curve: Suppose that  $m$  is even, and start at the South-West corner. Go East  $n - 1$  units, then North 1 unit. Next repeat the following movements  $m/2 - 1$  times: Go West  $n - 2$  units, North 1 unit, East  $n - 2$  units, and North 1 unit. Finally, go West  $n - 1$  units, and South  $m - 1$  units to close the curve.