THIRTY-SIXTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

Sponsored by
The Michigan Section of the Mathematical Association of America

Part I

October 14, 1992

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. **Check to insure that your six-digit code number has been recorded correctly.** Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please quit working immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgements. For each problem you should work out ideas on scratch paper before selecting the answer.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine an answer.

5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "**e)** none of these" or "**e)** none of the above". If you believe none of the first four alternatives to be correct, mark **e)** in such cases.

6. No one is permitted to explain to you the meaning of any question. Do not request anyone to break the rules of the competition. The use of books, tables, slide rules, electronic calculators, notes or any other aid is prohibited. If you have questions concerning the instructions, ask them now.

7. You many now open the test booklet and begin.

1. The cost of mailing a letter first class is 29 cents for the first ounce and 23 cents for each additional ounce. If the number of ounces is not a whole number, it is rounded up to the next larger whole number. The graph of the function $f(x)$, where $f(x)$ is the cost (in cents) of mailing a letter which weighs x ounces, most closely resembles

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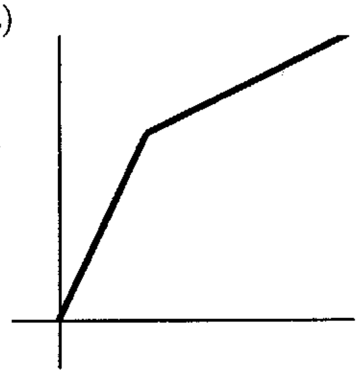
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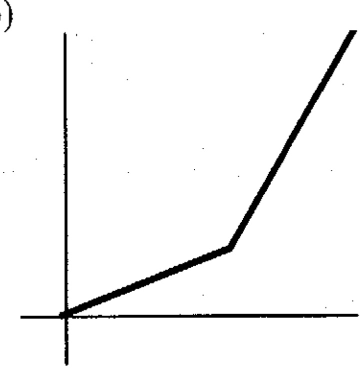
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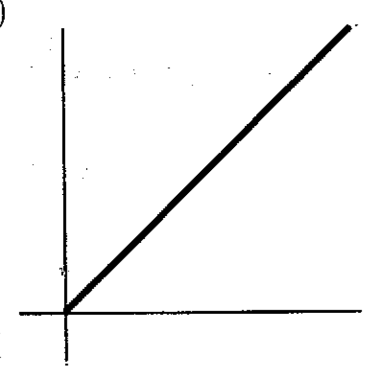

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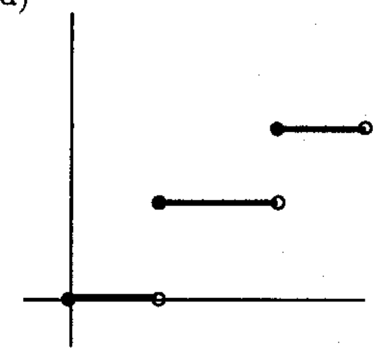
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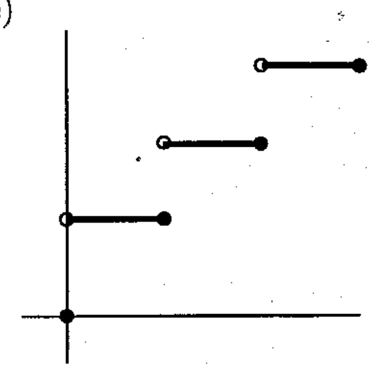
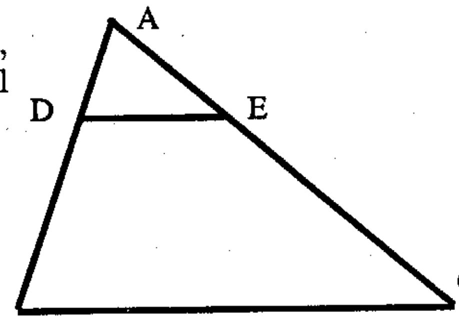


2. Find the point on the number line which is three-sevenths of the way from $2\frac{2}{3}$ to $7\frac{1}{3}$.

a) $4\frac{1}{3}$ b) $4\frac{4}{7}$ c) $4\frac{13}{21}$ d) $4\frac{2}{3}$ e) $4\frac{5}{7}$

3. The area of triangle ABC is equal to 1, $AB = 3AD$, and $AC = 3AE$. The area of the trapezoid $BDEC$ is equal to

a) 4/9 b) 5/9 c) 6/9 d) 7/9 e) 8/9



4. Express the solution of the system $\begin{array}{c} x - y = 6k \\ 2x + y = 3 \end{array}$ in terms of k.

a) $x = 2k$, $y = -4k$ b) $x = 2k + 1$, $y = 1 - 4k$ c) $x = 2k + 1$, $y = 3 - 4k$
d) $x = 2k + 3$, $y = 1 - 4k$ e) $x = 2k + 3$, $y = 3 - 4k$

5. Which of the following is NOT an identity?

a) $\sec(-x) = \sec x$ b) $\cos 2x = \cos^2 x - \sin^2 x$ c) $\sin^2(x/2) = \frac{1}{2}(1 - \cos x)$
d) $\log(x - y) = \dfrac{\log x}{\log y}$ e) $\log x^n = n \log x$

6. If $S = \dfrac{1}{1001} + \dfrac{1}{1002} + \dfrac{1}{1003} + \ldots + \dfrac{1}{10000}$, then .

a) $S < .09$ b) $.09 \leq S < .1$ c) $.1 \leq S < .9$ d) $.9 \leq S < 9$ e) $S \geq 9$

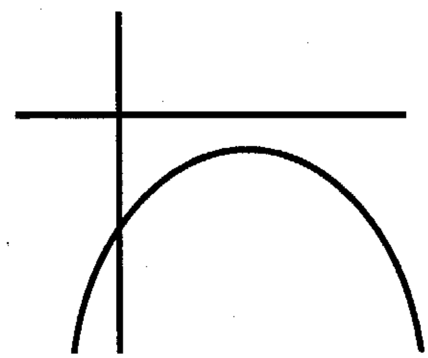
7. An isosceles triangle has a side of length 10 and a side of length 4. What is its perimeter?

a) 18 b) 20 c) 22 d) 24 e) Not enough information is given.

8. Find all integers x such that one of the statements $x^2 > 10$, $x^2 > 15$, $x^2 > 30$ is false and the other two are true.

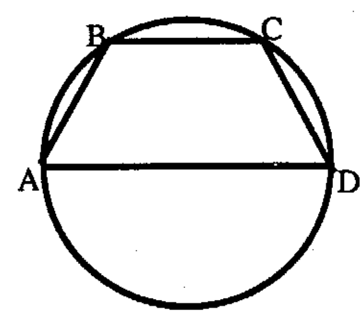
a) 6,7,8,9,... b) $0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ c) 4,5 d) $\pm 4, \pm 5$ e) None of these.

9. If the graph of $y = ax^2 + bx + c$ is as shown on the right, then which of the following is true?



a) $a > 0$, $b^2 - 4ac > 0$ b) $a > 0$, $b^2 - 4ac < 0$
c) $a < 0$, $b^2 - 4ac > 0$ d) $a < 0$, $b^2 - 4ac < 0$
e) Not enough information is given.

10. The base AD of trapezoid $ABCD$ is a diameter of a circle which passes through all the vertices of the trapezoid. Determine the area of the trapezoid if the radius of the circle equals 1 and the measure of angle ABC is 120°



a) 3/2 b) 3/4 c) $3\sqrt{3}$ d) $\frac{3\sqrt{3}}{2}$ e) None of these.

11. A deck of cards contains 26 red and 26 black cards, mixed randomly. If you divide the deck into two halves, a top half and a bottom half, of 26 cards each, then what is the probability that the top half will contain more black cards than the bottom half has red cards?

a) 0 b) 1/4 c) 1/2 d) 3/4 e) 1

12. Which of the following inequalities is depicted in this graph?

a) $|x+1|<1$ b) $|x-1|<3$ c) $|x-1|<2$ d) $|x+1|<3$ e) $|x+1|<2$

13. Solve the equation $S = P(10)^{2x}$ for x in terms of S and P.

a) $\frac{1}{2}(\log P - \log S)$ b) $\frac{\log S}{2\log P}$ c) $\frac{1}{10}\sqrt{\frac{S}{P}}$ d) $\frac{1}{2}\log\left(\frac{S}{P}\right)$ e) $\log\left(\frac{S}{.5P}\right)$

14. The volume of a given mass of gas varies directly as the temperature and inversely as the pressure. How is the volume affected if both the temperature and the pressure are doubled?

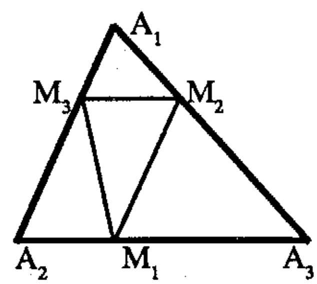
a) It is one-fourth as much. b) It is half as much. c) It remains the same.
d) It doubles. e) It is four times as much.

15. Some rectangles have lengths which are three units more than their widths. For such rectangles, express the perimeter P as a function of the length L.

a) $P = 4L - 6$ b) $P = 2L - 3$ c) $P = 6 + 4L$
d) $P = 3 + 2L$ e) $P = L(L-3)$

16. Suppose that $A_1A_2A_3$ is a triangle, that point M_1 is on side A_2A_3, that point M_2 is on side A_3A_1, and that point M_3 is on side A_1A_2. Suppose further that $A_1M_3 = \frac{1}{3}A_1A_2$, $A_2M_1 = \frac{1}{3}A_2A_3$, and $A_1M_2 = \frac{1}{3}A_1A_3$. What is the ratio of the area of triangle $A_1A_2A_3$ to the area of triangle $M_1M_2M_3$?

a) 3:1 b) 9:2 c) 9:4 d) 3:2 e) 4:1



17. Triangle I has sides of length 2, 3, and 9/2. Triangle II has angles equal to the angles of triangle I, a side of length 2, and a side of length 3 (not necessarily listed in the same order as the sides of triangle I). What are the possible lengths of the remaining side of triangle II?

a) 8/9 and 9/2 b) 9/2 c) 4/3 and 9/2 d) 7/6 and 9/2 e) 27/4 and 9/2

18. For any function $f(x)$ which takes real numbers to real numbers, define the function $\Delta(f)$ by $\Delta(f)(x) = f(x+1) - f(x)$. If $f(x) = x^2$, then $\Delta(\Delta(f))(x)$ is equal to

a) 2 b) $2x+1$ c) 0 d) x^2 e) 1

19. Paul is 6 feet tall and is walking away from a street light having a lamp 15 feet above the ground. Express the length, y, of his shadow on the ground as a function of the horizontal distance, x, between Paul and the light pole.

a) $y = \frac{2}{3}x$ b) $y = \frac{1}{3}x$ c) $y = \frac{1}{4}x$ d) $y = \frac{3}{5}x$ e) $y = \frac{2}{5}x$

20. The solutions to the equation $x^2 + \dfrac{x}{1992 \cdot 1993} - \dfrac{1}{1992 \cdot 1993} = 0$ are

a) $\dfrac{1}{1992}$ and $\dfrac{1}{1993}$ b) $\dfrac{-1}{1992}$ and $\dfrac{1}{1993}$ c) $\dfrac{1}{1992}$ and $\dfrac{-1}{1993}$ d) $\dfrac{-1}{1992}$ and $\dfrac{-1}{1993}$
e) None of these.

21. One has to distribute four different toys among three children; each child must get one or two toys. How many possible ways are there to do this?

a) 3 b) 6 c) 12 d) 24 e) 36

22. What is the period of the function $f(x) = \tan(x) - \cot(x)$?

a) $\pi/4$ b) $\pi/2$ c) π d) $3\pi/2$ e) 2π

23. Let α be the measure (in degrees) of an obtuse angle. If $\sin\alpha = 0.7$, then

a) $90° < \alpha \le 120°$ b) $120° < \alpha \le 135°$ c) $135° < \alpha \le 150°$
d) $150° < \alpha \le 180°$ e) None of these.

24. Which of the following is equivalent to $\dfrac{x}{3x+2} > 1$?

a) $x < -1$ b) $x > -1$ c) $-1 < x < -2/3$
d) $x < -1$ or $x > -2/3$ e) None of these.

25. Suppose that $a > b$ and $c > d$. Which is greater, $ac + bd$ or $ad + bc$?

a) $ac + bd$, always. b) $ad + bc$, always.
c) $ac + bd$ if a, b, c, and d are all positive and $ad + bc$ if a, b, c, and d are all negative.
d) $ad + bc$ if a, b, c, and d are all positive and $ac + bd$ if a, b, c, and d are all negative.
e) None of these.

26. Suppose N is an even integer, $N + 1$ is divisible by 3, and $N + 5$ is divisible by one of the following five numbers. Which one?

a) 2 b) 3 c) 4 d) 5 e) 6

27. The intersection of a cube and a plane can never be

a) a triangle b) a rectangle c) a rhombus
d) a quadrilateral without parallel sides e) a pentagon

28. Which of the following numbers is not a root of the polynomial $6x^4 + 49x^3 + 128x^2 + 132x + 45$?

a) -1.5 b) 1.5 c) $\frac{-5-\sqrt{13}}{2}$ d) $\frac{-5+\sqrt{13}}{2}$ e) $-\frac{5}{3}$

29. Find the *number* of solutions to the equation $\sin 2x = \frac{\pi}{3}$ in the interval $0 < x < 2\pi$.

a) none b) one c) two d) three e) four

30. The graph of a function passes through the points $(1,3)$, $(2,-5)$, $(3,0)$, and one of the following five points. Which one?

a) $(4,-5)$ b) $(2,6)$ c) $(3,3)$ d) $(1,0)$ e) $(2,0)$

31. Let O, P, and Q be circles of radius one, positioned so that O and Q are externally tangent, and the center of P is at the point of tangency of O and Q. Let X be the area of the region inside O but outside P, and let Y be the area of the region inside P but outside both O and Q. Then $X - \frac{1}{2}Y =$

a) $\pi/4$ b) $\pi/2$ c) π d) 1 e) None of these.

32. Suppose that the following definitions are agreed upon:
A *vertical line* is a line containing the center of the earth.
A *horizontal line* is a line perpendicular to some vertical line.
Which of the following statements are true?
I. Every line is horizontal.
II. Every vertical line is a horizontal line.
III. Every horizontal line is a vertical line.

a) None are true. b) II only c) I and II only
d) II and III only e) I, II, and III

33. Find the *number* of solutions to the system $\begin{matrix} y = x^2 - 5 \\ x^2 + y^2 = 25 \end{matrix}$.

a) none b) one c) two d) three e) four

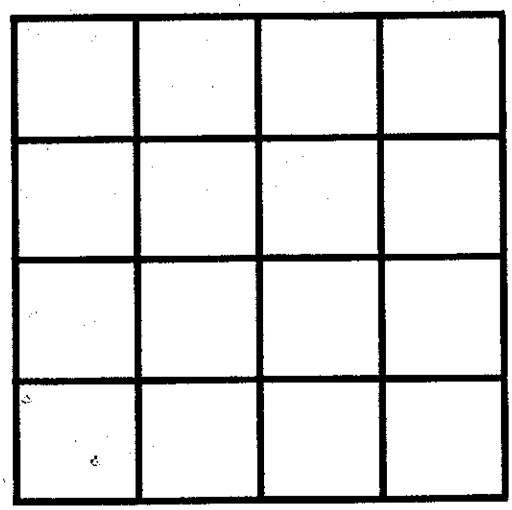
34. Express the area of a square as a function of the length x of one of its diagonals.

a) $A = \dfrac{x^2}{4}$ b) $A = \dfrac{x^2}{2}$ c) $A = \dfrac{x^2}{\sqrt{2}}$ d) $A = \sqrt{2}x^2$ e) $A = 2x^2$

35. Consider 71 consecutive integers. If 15 of them are divisible by 5 and 11 of them are divisible by 7, then how many of them are divisible by 35?

a) 0 b) 1 c) 2 d) 3 e) 4

36. How many squares, of any size, are in this diagram?



a) 16 b) 25 c) 30 d) 32 e) 64

37. A hat contains 2 red marbles, 3 blue marbles, and 4 green marbles. You choose two different marbles at random. What is the probability that they are the same color?

a) 1/2 b) 5/18 c) 61/144 d) 29/81 e) None of these.

38. Sides AB and AC of triangle ABC are the diameters of two circles. If X is the intersection point of these circles other than A, then point X

a) must be in the interior of triangle ABC
b) must be outside triangle ABC
c) must be on the line containing the segment BC
d) cannot be on the line containing the segment BC
e) must be either inside or outside triangle ABC

39. If $x = 7^7$ and $y = x^x$ then $\sqrt[7]{y} =$

a) x^7 b) 7^x c) 7^{7x} d) x^{7x} e) None of these.

40. Arthur is as old now as Betty was when Charlie was as old as Betty will be when Arthur will be as old as Betty is now. If Charlie is now twice as old as Betty, what is the ratio of Betty's current age to Arthur's current age?

a) 1:2 b) 1:1 c) 2:1 d) 3:1 e) Not determined.

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