THIRTY-FIFTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

Sponsored by
The Michigan Section of the Mathematical Association of America

Part I

October 9, 1991

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. **Check to insure that your six-digit code number has been recorded correctly.** Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please quit working immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgements. For each problem you should work out ideas on scratch paper before selecting the answer.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine an answer.

5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "e) none of these" or "e) none of the above". If you believe none of the first four alternatives to be correct, mark e) in such cases.

6. No one is permitted to explain to you the meaning of any question. Do not request anyone to break the rules of the competition. The use of books, tables, slide rules, electronic calculators, notes or any other aid is prohibited. If you have questions concerning the instructions, ask them now.

7. You many now open the test booklet and begin.

1. Find the distance between the solutions of $| 4x + 2 | = 10$.

   a) 1   b) 2   c) 3   d) 4   e) 5

2. The graph of the equation $y = -\frac{5}{3} x - 4$ does not appear in quadrant(s)

   a) I   b) II   c) III   d) IV   e) I and IV

3. Find the value of k which will make the following system inconsistent:

   $$4x - 6y = -2$$
   $$kx - 9y = 3$$

   a) 2   b) -3   c) 3   d) -6   e) 6

4. The minute hand of a clock is 3 inches long. Through what distance
   (nearest tenth) does its tip move in 15 minutes?

   a) 4.5 in   b) 4.7 in   c) 4.9 in   d) 5.1 in   e) 5.3 in

5. If $100^n = 1991$, then n =
   a) $\log 1991 - 2$ b) $\log 19.91$ c) 19.91   d) $\sqrt[100]{1991}$

   e) $\frac{\log 1991}{2}$

6. Strips of plastic of length x are bent into circles. Find the area
   inside each circle as a function of x.

   a) $A = \frac{x^2}{\pi}$ b) $A = \frac{x^2}{2\pi}$ c) $A = \frac{x^2}{4\pi}$ d) $A = \pi x^2$ e) $A = \frac{x^2}{4}$

7. How many liters of a 10% alcohol solution must be mixed with 30 liters of
   a 50% solution to get a 20% solution?

   a) 50   b) 60   c) 80   d) 90   e) 120

8. Fred and John want to paint Fred's motorboat. Fred can do it in 3 hours
   and John can do it in 6 hours. How long will it take them working
   together.

   a) 1 hour   b) $1\frac{1}{2}$ hours   c) 2 hours   d) $2\frac{1}{2}$ hours   e) $4\frac{1}{2}$ hours

9. Mary averages 12 miles per hour riding her bike to school. Averaging 36 miles per hour by car takes her one-half hour less time. How far does she travel to school?

a) 12 miles   b) 9 miles   c) 15 miles   d) 20 miles   e) 36 miles

10. The figure to the right shows a circle with center O where OA = AB. If the radius of the circle is 8 then the ratio of the arc length BC to the arc length CD is

a) $\dfrac{\sqrt{3}}{2}$ b) 1   c) $\sqrt{3}$ d) 2   e) $\dfrac{3}{2}$
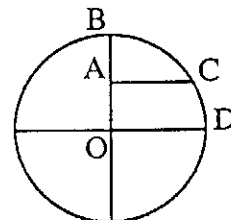
11. A circular island of diameter 100 yards is surrounded by a concentric circular moat whose area equals that of the island. Which of the following best approximates the width of the moat?

a) 20 yards   b) 30 yards   c) 50 yards   d) 70 yards   e) 100 yards

12. In a cube whose sides have length 1, join two diagonally opposite corners by a line segment (passing through the cube's center). How long is this segment?

a) $\sqrt{2}$ b) $\sqrt{3}$ c) 2   d) $2\sqrt{2}$ e) none of these

13. For any real number x such that $|x| \geq 1$, the expression $\sec^2(\tan^{-1}x) - \tan^2(\sec^{-1}x)$ is equal to

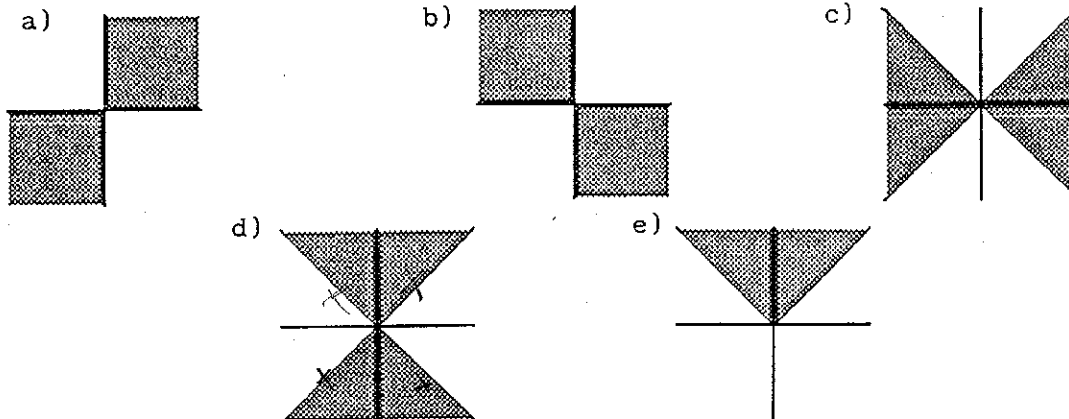a) 1   b) 2   c) 2x   d) x^2 e) $2\sqrt{x^2+1}$

14. A test has two sections of 5 questions each. The instructions say to answer 3 questions in each section. In how many ways can a student choose the questions to answer in order to complete the exam?

a) 15   b) 25   c) 50   d) 100   e) none of these

15. The graph of the equation $y = e^{\ln(x)}$ most closely resembles

a)   b)   c)

d)   e)
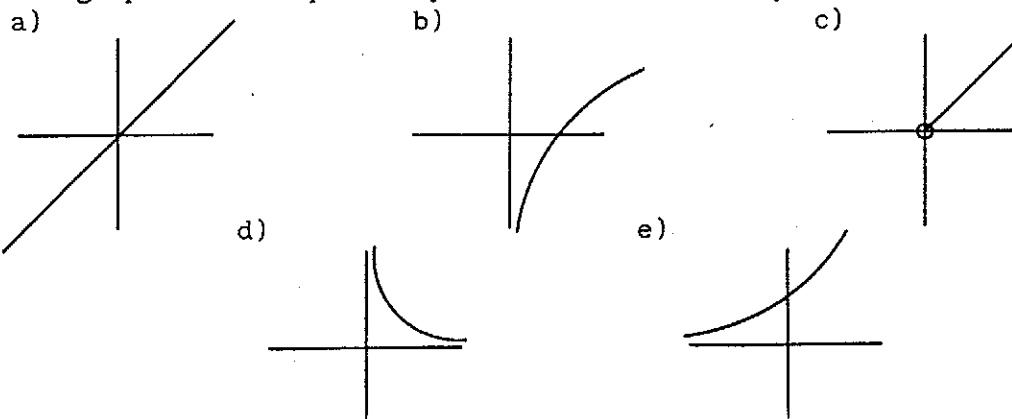
16. The graph of the inequality $|x+y| \geq |x-y|$ is

a)

b)

c)

d)

e)

17. What is the product of the divisors of 1000 (including 1 and 1000)?

   a) 10^{24} b) 10^{16} c) 10^8 d) 10^{30} e) 10^{32}

18. In which interval can a solution to $2^x - 2^{-x} = 1000$ be found?

   a) $x < -20$ b) $-20 < x < -5$ c) $-5 < x < 5$
   d) $5 < x < 20$ e) $x > 20$

19. Find a factor of $9u^2 + 16v^2 - 4w^2 - 24uv$.

   a) $3u + 4v - 2w$ b) $3u - 4v - 4w$ c) $3u - 2v - 2w$
   d) $3u + 2v - 4w$ e) $3u - 4v - 2w$

20. Given $\log_b 2 = x$, $\log_b 3 = y$, evaluate $\log_b 6\sqrt{2}$

   a) $\sqrt{x}\, y$ b) $\sqrt{x}\, xy$ c) $\frac{3}{2}x + y$ d) $\frac{1}{2}x + y$ e) $\frac{3}{2}xy$

21. Find the range of the function $f(x) = a + 4 \sin (3x + b)$, where a and b are constants.

   a) $[a - 4, a + 4]$ b) $[4 - a, 4 + a]$ c) $\left[\frac{-b}{3}, \frac{-b}{3} + 2\pi\right]$

   d) $[a - 12, a + 12]$ e) $\left[\frac{-b}{3}, \frac{-b}{3} + \frac{2\pi}{3}\right]$

22. Express the area A of an equilateral triangle as a function of its perimeter p.

   a) $A = \dfrac{\sqrt{3}}{36} p^2$ b) $A = \dfrac{1}{18} p^2$ c) $A = \dfrac{\sqrt{3}}{12} p^2$

   d) $A = \dfrac{1}{9} p^2$ e) $A = \dfrac{\sqrt{3}}{4} p^2$

23. Which of the following equations has a graph that looks most nearly like the figure?
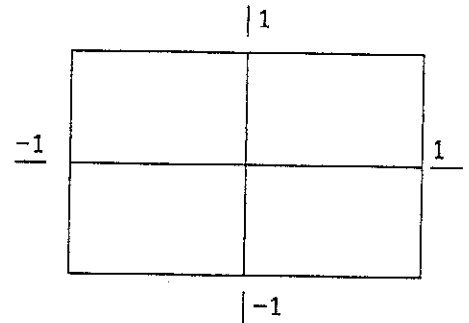
   a) $x^{99} + y^{99} = 1$ b) $x^{100} + y^{100} = 1$

   c) $x^{\frac{1}{99}} + y^{\frac{1}{99}} = 1$ d) $x^{\frac{1}{100}} + y^{\frac{1}{100}} = 1$

   e) $|x| + |y| = 1$

24. What proportion of the area of a circle lies inside an inscribed regular hexagon?

   a) $\dfrac{3\sqrt{3}}{4\pi}$ b) $\dfrac{3\sqrt{3}}{2\pi}$ c) $\dfrac{3}{\pi}$

   d) $\dfrac{3\sqrt{3}}{\pi}$ e) depends on the size of the circle

25. A solution of the equation $\log (x - 4) - \log (3x - 10) = \log \dfrac{1}{x}$ is

   a) 4   b) 5   c) 6   d) 7   e) 8

26. The term that does not contain x in the expansion of $\left(6x - \dfrac{1}{2x}\right)^6$ is

   a) 540   b) -27   c) 27   d) -540   e) there is no such term

27. A boy on a bicycle coasts down a hill, covering 4 feet the first second and in each succeeding second 5 feet more than in the preceding second. If he reaches the bottom of the hill in 11 seconds, find the total distance (in feet) traveled.

   a) 265   b) 283   c) 319   d) 337   e) 378

28. How many integers from 100 to 999 can be written with no 7's?

     a) 512   b) 648   c) 729   d) 799   e) 899

29. A survey of movie lovers produced the following results. 30 like war
     movies, 42 like love stories, 55 like comedies, 4 like war movies and
     love stories, 20 like war movies and comedies, 28 like love stories and
     comedies, 2 like all three, and 22 like none of the three. What is the
     number of people surveyed?

     a) 99   b) 100   c) 105   d) 127   e) 203

30. You are given three mutually externally tangent circles, each of radius
     1. Three lines, each tangent to two of the circles and not touching the
     third, form an equilateral triangle. What is the length of a side of
     this triangle?

     a) $3+\sqrt{3}$ b) 6   c) $1+\sqrt{3}$ d) $2\sqrt{3}$ e) $2+2\sqrt{3}$

31. Each side of square $ABCD$ has length 1. Let K,L,M,N be the midpoints of
     sides AB,BC,CD,DA, respectively. Let X be the intersection point of BN
     and DK, and let Y be the intersection point of BM and DL. Find the area
     of the diamond-shaped region $BXDY$.

     a) $\dfrac{1}{2}$ b) $\dfrac{\sqrt{2}}{2}$ c) $\dfrac{\sqrt{3}}{4}$ d) $\dfrac{1}{3}$ e) $\dfrac{1}{4}$

32. An isosceles trapezoid which is circumscribed about a circle has two
     adjacent angles of $120°$and two of $60°$. What is the ratio of its smaller
     base to its greater base?

     a) $\sqrt{3}:2$ b) $\sqrt{2}:2$ c) 1:3   d) 2:3   e) $\sqrt{2}:3$

33. Each side of square $ABCD$ has length 1. Isosceles right triangles ADE
     and BCF, pointing outward from the square, have hypotenuses AD and BC,
     respectively. An ellipse passes through points A,B,C,D,E and F.
     What is the length of its minor axis?

     a) 2   b) $\sqrt{2}$ c) $\dfrac{\sqrt{3}}{2}$ d) 1   e) $\dfrac{2}{\sqrt{3}}$
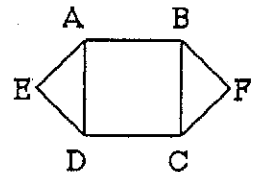
34. How many real numbers x such that $0 < x < \pi$ are solutions to the equation
     $\sin\left(\dfrac{1}{x}\right) = 0$?

     a) 0   b) 1   c) 2   d) 3   e) infinitely many

35. A hat contains one chip marked with the number 1, two chips marked with the number 2, three chips marked with the number 3, and so on, through one hundred chips marked with the number 100. You choose one chip at random from the hat. What is the probability that it is marked with an even number?

    a) $\dfrac{1}{2}$ b) $\dfrac{51}{101}$ c) $\dfrac{51}{100}$ d) $\dfrac{50}{99}$ e) $\dfrac{49}{99}$

36. Suppose that $11 = 49a - 7b + c$, where each of a, b and c is one of the numbers 0,1,2,3,4,5,6. What is b?

    a) 0   b) 1   c) 2   d) 5   e) 6

37. Find the number of distinguishable 3-letter sequences that can be made using the letters in the word MICHIGAN.

    a) 120   b) 210   c) 228   d) 498   e) 3360

38. Which of the following statements are true about a quadrilateral?

    I. If the diagonals bisect each other, the quadrilateral must be a parallelogram.
    II. If two opposite angles are right angles, then the quadrilateral must be a rectangle.
    III. If the diagonals are equal in length and perpendicular, then the quadrilateral must be a square.

    a) None   b) I only   c) I and II only   d) I and III only
    e) I, II, and III

39. The legs of a right triangle are 8 and 15. Find the altitude to the hypotenuse.

    a) $6\dfrac{15}{17}$ b) $6\dfrac{16}{17}$ c) 7   d) $7\dfrac{1}{17}$ e) $7\dfrac{2}{17}$

40. Three distinct numbers are chosen at random from the set {0,1,2,3,4,5,6,7,8,9,10}. What is the probability that the second-largest of them is 5?

    a) 1/45   b) 1/10   c) 1/2   d) 5/33   e) 7/26

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Assocation of America

DIRECTOR

Ruth G. Favro
Lawrence Technological University

**OFFICERS OF THE
MICHIGAN SECTION
COMMITTEE**

Chairperson

Renate McLaughlin
University of Michigan - Flint

Vice Chairpersons

Melvin A. Nyman
Alma College

Lowell C. Stultz
Kalamazoo Valley Comm. College

Secretary-Treasurer

Thomas J. Miles
Central Michigan University

Governor

Elliot Tanis
Hope College

EXAMINATION

Chairperson

Andreas Blass
University of Michigan

Ahmed Assaf
Central Michigan University

Paul J. Eenigenburg
Western Michigan University

Kenneth Schilling
University of Michigan - Flint

ACKNOWLEDGEMENTS

The Michigan Association of Secondary School Principals has placed this competition on the Approved List of Michigan Contests and Activities.