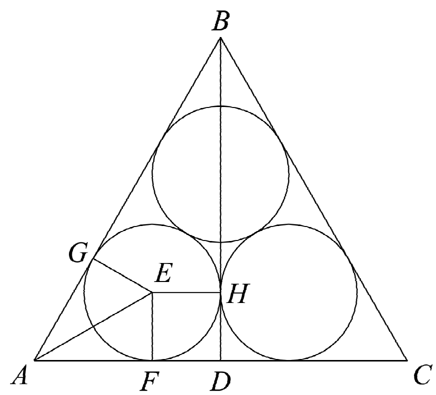
Michigan Mathematics Prize Competition, 2013, Part II — Solutions

1. The number 100 is written as a sum of distinct positive integers. Determine, with proof, the maximum number of terms that can occur in the sum.

Solution. The maximum number of summands is 13. To express 100 as a sum of 13 distinct positive integers we write $100 = 22 + 1 + 2 + \cdots + 12$. On the other hand, a sum of 14 distinct positive integers is at least as large as $1 + 2 + \cdots + 14 = 14 \times 15/2 = 105 > 100$, so 100 cannot be represented as a sum of 14 or more distinct positive integers.

2. Inside an equilateral triangle of side length s are three mutually tangent circles of radius 1, each one of which is also tangent to two sides of the triangle, as depicted below. Find s.



Solution. Label the figure as shown. The triangles $\triangle AEF$, $\triangle AEG$ are both right triangles, with a common hypotenuse, and a leg of length 1, so by the Pythagorean theorem the other legs, AF and AG, also have the same length. Hence the two triangles are congruent, and in particular angles $\angle FAE$ and $\angle GAE$ have the same measure. Since $\angle BAC$ has measure $60°$, we see that $\angle FAE$ has measure $30°$, and so $\triangle EFA$ is a 30-60-90 right triangle. The edges therefore have lengths proportional to $1/2 : \sqrt{3}/2 : 1$. Since the short leg EF has length 1, it follows by similarity that the longer leg AF has length $\sqrt{3}$. The segment FD is parallel to the segment EH, and has the same length, so FD has length 1. Hence AD has length $1 + \sqrt{3}$. Since this is $s/2$ we conclude that $s = 2 + 2\sqrt{3}$.

3. Color a 4×7 rectangle so that each of its 28 unit squares is either red or green. Show that no matter how this is done, there will be two columns and two rows, so that the four squares occurring at the intersection of a selected row with a selected column all have the same color.

First proposed solution. If there are two columns (of height 4) with 3 or more green squares in them, then there will be two rows that have green squares in those columns, since two subsets of size at least 3 in a set of 4 elements must have an intersection of size at least 2. Thus we may suppose that there is at most one such column, and similarly at most one column with 3 or more red squares. Suppose every column has 2 red squares and 2 green squares. The number of subsets of size 2 of a set of 4 elements is $\binom{4}{2} = 6$. Since there are 7 columns, there must be a subset of rows that occurs twice. Suppose that there are 5 columns with 2 green and 2 red squares, and one column containing 3 or more green squares. From the 3 rows with green squares in that column, we can choose 2 in 3 different ways. Thus a total of 8 subsets of 2 rows arise, and hence the same pair must occur at least twice. The argument is similar in the case that there is a column with 3 or more red squares.

A second possible solution. We arrange the rectangle so that it has 4 rows and 7 columns. We form 6 green boxes labeled $g_{1,2}$, $g_{1,3}$, $g_{1,4}$, $g_{2,3}$, $g_{2,4}$, $g_{3,4}$ and 6 red boxes labeled $r_{1,2}$, $r_{1,3}$, $r_{1,4}$, $r_{2,3}$, $r_{2,4}$, $r_{3,4}$. Consider column k. If in rows i and j both squares are green, then place the number k in box $g_{i,j}$. Do this for all pairs $i < j$, all columns k, $1 \leq k \leq 7$, and both colors. If in column k all squares are green, then the number k is entered in all 6 green boxes. If in column k there are 3 green squares and one red square, then k is entered in 3 green boxes. If in column k there are two green squares and two red squares, then the number k is entered in one green box and one red box. If in column k there is one green square and three red squares, then k is entered in three red boxes. If in column k all squares are red, then the number k is entered in all six red boxes. Note that in all five of these cases, the number k is entered in at least two boxes. Thus when this is done for all seven columns, at least fourteen numbers have been put in the boxes. Since there are only twelve boxes, there must be a box with at least two numbers in it. Suppose that the numbers k and ℓ with $k \neq \ell$ are in box $g_{i,j}$. Then all four vertices of the rectangle $(i,k), (j,k), (j,\ell), (i,\ell)$ are green. Similarly, if $r_{i,j}$ contains two numbers, then we have a rectangle whose vertices are all red.

4. (a) Show that the y-intercept of the line through any two distinct points of the graph of $f(x) = x^2$ is -1 times the product of the x-coordinates of the two points.

(b) Find all functions with the property that the y-intercept of the line through any two distinct points of its graph is -1 times the product of the x-coordinates. Prove that you have found all such functions and that all functions you have found have this property.

Solution. (a) The line through two points (x_1, y_1), (x_2, y_2) with $x_1 \neq x_2$ is given by the equation

$$y = y_1 \frac{x - x_2}{x_1 - x_2} + y_2 \frac{x - x_1}{x_2 - x_1}\,. \tag{1}$$

We take $y_1 = x_1^2$, $y_2 = x_2^2$ and $x = 0$ to find that the y-axis intercept is at

$$y = x_1^2 \frac{-x_2}{x_1 - x_2} + x_2^2 \frac{-x_1}{x_2 - x_1} = -x_1 x_2\,.$$

(b) Suppose that f is a function with the desired property. We take $(x_1, y_1) = (1, f(1))$ and $(x_2, y_2) = (x_2, f(y_2))$ where $x_2 \neq 1$. By (1) the line through these points is

$$y = f(1)\frac{x - x_2}{1 - x_2} + f(x_2)\frac{x - 1}{x_2 - x_1}\,.$$

If this is $-x_2$ when $x = 0$, then

$$-x_2 = f(1)\frac{-x_2}{1 - x_2} + f(x_2)\frac{-1}{x_2 - 1},$$

which on rearranging gives

$$f(x_2) = x_2^2 + (f(1) - 1)x_2$$

for all $x_2 \neq 1$. But this also holds for $x_2 = 1$, so $f(x) = x^2 + cx$ for all x where $c = f(1) - 1$.

Conversely, if $f(x) = x^2 + cx$ for all x, then by (1) with $y_1 = x_1^2 + cx_1$, $y_2 = x_2^2 + cx_2$, and $x = 0$ we find that the y-intercept is

$$y = (x_1^2 + cx_1)\frac{-x_2}{x_1 - x_2} + (x_2^2 + cx_2)\frac{-x_1}{x_2 - x_1} = -x_1x_2,$$

as desired. Thus the set of functions in question is precisely the functions of the form $f(x) = x^2 + cx$ where c is a constant.

5. Let n be a positive integer. We consider sets $\mathcal{A} \subseteq \{1, 2, \ldots, n\}$ with the property that the equation $x + y = z$ has no solution with $x \in \mathcal{A}$, $y \in \mathcal{A}$, $z \in \mathcal{A}$.

(a) Show that there is a set $\mathcal{A}$ as described above that contains $[(n+1)/2]$ members where $[x]$ denotes the largest integer less than or equal to x.

(b) Show that if $\mathcal{A}$ has the property described above, then the number of members of $\mathcal{A}$ is less than or equal to $[(n+1)/2]$.

Solution. (a) Take $\mathcal{A}$ to consist of all odd numbers a such that $1 \leq a \leq n$. Then $\mathcal{A}$ contains $[(n+1)/2)]$ elements, and if $a \in \mathcal{A}$, $a' \in \mathcal{A}$, then $a + a' \notin \mathcal{A}$ because $a + a'$ is even. Alternatively, take $\mathcal{A}$ to consist of all integers a such that $n/2 < a \leq n$. Then $\mathcal{A}$ contains $[(n+1)/2]$ elements, and $a + a' \notin \mathcal{A}$ because $a + a' > n$.

(b) Let a_1 be the least member of $\mathcal{A}$, and set

$$\mathcal{B} = \{a \in \mathcal{A} \,:\, a_1 \leq a \leq n - a_1\}, \qquad B = \operatorname{card}\mathcal{B},$$
$$\mathcal{C} = \{a \in \mathcal{A} \,:\, n - a_1 < a \leq n\}, \qquad C = \operatorname{card}\mathcal{C}\,.$$

Let $a_1 + \mathcal{B} = \{a_1 + b \,:\, b \in \mathcal{B}\}$. Note that if $b \in \mathcal{B} \subseteq \mathcal{A}$, then $a_1 + b \notin \mathcal{A}$. The sets $\mathcal{B}$, $a_1 + \mathcal{B}$, $\mathcal{C}$ are pairwise disjoint and lie in the interval $[a_1, n]$. Hence

$$2B + C \leq n - a_1 + 1\,.$$

The set $\mathcal{C}$ is a subset of the interval $(n - a_1, n]$, and so

$$C \leq a_1\,.$$

On adding these inequalities we see that

$$2\text{card}\,\mathcal{A} = 2B + 2C \leq n + 1,$$

so $\text{card}\,\mathcal{A} \leq (n+1)/2$. Since $\text{card}\,\mathcal{A}$ is an integer, it follows that $\text{card}\,\mathcal{A} \leq [(n+1)/2]$.