MMPC STUDENT CODE

FIFTY-SEVENTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America

Part II

December 11, 2013

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**

2. Part II consists of problems and proofs. You will be allowed 100 minutes (1 hour and 40 minutes) for the five questions. To receive full credit for a problem, you are expected to justify your answer.

3. You are not expected to solve all the problems completely. Look over all the problems and work first on those that interest you the most. If you are unable to solve a particular problem, partial credit might be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.

4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the blank pages at the end of the booklet (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page clearly write the **problem number**. If you use additional paper for your answer, check the appropriate box and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.

5. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.

6. The competition rules prohibit you from asking questions of anyone during the examination. The use of notes, reference material, computation aids, or any other aid is likewise prohibited. Please note that **calculators are not allowed** on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. Please do not return scratch paper containing routine numerical calculations.

7. You may now open the test booklet.

#	1	2	3	4	5	Total
Score						

1. The number 100 is written as a sum of distinct positive integers. Determine, with proof, the maximum number of terms that can occur in the sum.

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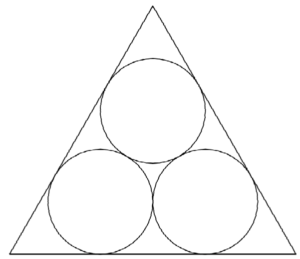
2. Inside an equilateral triangle of side length s are three mutually tangent circles of radius 1, each one of which is also tangent to two sides of the triangle, as depicted below. Find s.



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3. Color a 4×7 rectangle so that each of its 28 unit squares is either red or green. Show that no matter how this is done, there will be two columns and two rows, so that the four squares occurring at the intersection of a selected row with a selected column all have the same color.

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4. (a) Show that the y-intercept of the line through any two distinct points of the graph of $f(x) = x^2$ is -1 times the product of the x-coordinates of the two points.

   (b) Find all real valued functions with the property that the y-intercept of the line through any two distinct points of its graph is -1 times the product of the x-coordinates. Prove that you have found all such functions and that all functions you have found have this property.

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5. Let n be a positive integer. We consider sets $\mathcal{A} \subseteq \{1, 2, \ldots, n\}$ with the property that the equation $x + y = z$ has no solution with $x \in \mathcal{A}$, $y \in \mathcal{A}$, $z \in \mathcal{A}$.

   (a) Show that there is a set $\mathcal{A}$ as described above that contains $[(n+1)/2]$ members where $[x]$ denotes the largest integer less than or equal to x.

   (b) Show that if $\mathcal{A}$ has the property described above, then the number of members of $\mathcal{A}$ is less than or equal to $[(n+1)/2]$.

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(continued solutions)

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

DIRECTOR
Stephanie Edwards
Hope College

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We wish to thank Hope College and Mu Alpha Theta for their support of this competition.

The Michigan Association of Secondary School Principals has placed this competition on the Approved List of Michigan Contests and Activities.