THE SIXTY-THIRD ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

Sponsored by
The Michigan Section of the Mathematical Association of America

Part I

Tuesday, October 8, 2019

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Carefully read and follow the instructions printed on the answer sheet. Check to ensure that your five-digit MMPC code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor asks you to stop, please quit working immediately and turn in your answer sheet.

3. Consider the problems and responses carefully. You may work out ideas on scratch paper before selecting a response.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.

5. For each of the questions, five different possible responses are provided. Choose the correct answer and completely fill in the corresponding bubble on your answer sheet.

6. Any scientific or graphing calculator is permitted on Part I. Unacceptable machines include computers, PDAs, pocket organizers, cell phones, and similar devices. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator. No other devices are permitted.

7. No one is permitted to explain to you the meaning of any question. Do not ask anyone to violate the rules of the competition. If you have questions concerning the instructions, ask them now.

8. You may open the test booklet and begin.

1. Suppose that A, B, and C are sets such that $|A| = 16$, $|A \cap C| = 4$, $|A \cap B| = 7$, and $|A \cap B \cap C| = 3$. Find the number of elements that are in A only.

(A) 2 (B) 8 (C) 9 (D) 11 (E) 12

2. How many of the fractions

$$\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{19}, \frac{1}{20}$$

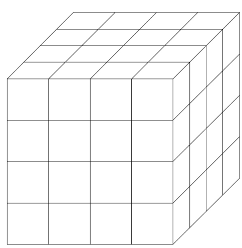
have a non-trivial repeating decimal expansion? (A repeating expansion where the repeating part consists of 0's or 9's is considered trivial, like in $\frac{1}{2} = 0.5000\cdots = 0.4999\ldots$)

(A) 10 (B) 11 (C) 12 (D) 13 (E) 14

3. How many ordered pairs (a, b) of positive integers are there such that $a \leq 5 \leq b$ and $a, 5, b$ are the lengths of three sides of a triangle?

(A) 10 (B) 15 (C) 20 (D) 25 (E) 30

4. A transparent plastic cube is painted red on all of its six sides, and then is divided into 64 equal-sized small cubes as illustrated below:



Randomly pick a small cube, such that each small cube is equally likely to be picked. What is the probability that the small cube has red paint exactly on two sides?

(A) $\frac{1}{8}$ (B) $\frac{3}{16}$ (C) $\frac{1}{4}$ (D) $\frac{3}{8}$ (E) $\frac{1}{2}$

5. Consider a sequence $a_1, a_2, \ldots$ such that $a_{n+1} = a_1 + 2a_2 + 3a_3 + \ldots + na_n$, for all $n = 1, 2, \ldots$. (In other words, $a_2 = a_1$, $a_3 = a_1 + 2a_2$, $a_4 = a_1 + 2a_2 + 3a_3$, and so on.) If $a_{2019} = 2019$, find the value of a_{2018}.

(A) 1 (B) 2018 (C) 2019 (D) $\frac{2019}{2}$ (E) $\frac{2019}{2018}$

6. Denote by x_1, x_2, and x_3 the roots of $x^3 = 1$. Then, for all counting numbers n, the quantity $x_1^n + x_2^n + x_3^n$ equals ...

(A) 0 (B) 3 (C) $(x_1 + x_2 + x_3)^n$ (D) $x_1^n x_2^n + x_1^n x_3^n + x_2^n x_3^n$ (E) $x_1^n x_2^n x_3^n$

7. Let ABC be a right triangle with hypotenuse c and legs a and b. Draw an altitude from the right angle to the hypotenuse. Which of the following formulae gives the correct expression for the length of this altitude?

(A) $\dfrac{a^{-1} + b^{-1}}{ab}$ (B) $\dfrac{\sqrt{ab}}{c}$ (C) $\dfrac{ab}{c}$ (D) $\dfrac{c}{ab}$ (E) $a^{-1} + b^{-1} + c^{-1}$

8. Which of the following are rational numbers (numbers which can be expressed as the ratio of two integers)?

I. The only negative root of $p(x) = x^3 - 3x + 1$.

II. $\log_{35}(1 + 2 + \cdots + 49)$.

III. The only positive root of $q(x) = x^3 - 15x - 50$.

(A) II (B) III (C) I and II (D) I and III (E) II and III

9. Let $x \vee y$ denote the larger of the values x and y and $x \wedge y$ denote the smaller of the values x and y. Suppose that we know $v < w < x < y < z$. What is

$$(((y \vee z) \wedge v) \vee ((v \vee x) \wedge w))) \wedge ((z \wedge y) \vee (w \vee y))?$$

(A) v (B) w (C) x (D) y (E) z

10. A robot has to move n ft, where n is a positive integer, in a straight direction. The robot is allowed two types of moves. A Type I move is where the robot moves 1 ft. A Type II move is where the robot moves 2 ft. Let R_n be the number of ways the robot can move n ft. Compute R_{10}.

(A) 45 (B) 68 (C) 88 (D) 89 (E) 90

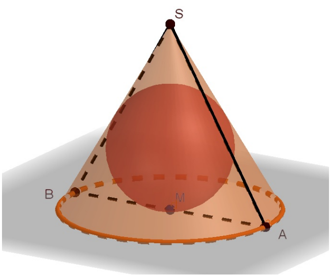
11. Consider the sequence $a_1, a_2, \ldots,$ such that $a_1 = 1$ and $a_n = 2a_{n-1}$, for all integers $n > 1$. Compute the value of $\sum_{i=1}^{2019}(a_{i+1} - a_i)$.

(A) $2^{2019} - 1$ (B) 2^{2019} (C) $2^{2020} - 1$ (D) 2^{2020} (E) $2^{2021} - 1$

12. Suppose that you have a deck of n cards numbered 1 through n, with n at least 3. Shuffle these cards so that the deck is in random order. What is the probability that the card labeled 1 is in the first, or $\lfloor \frac{n+1}{2} \rfloor$, or n position? (Here $\lfloor x \rfloor$ denotes the floor of x, that is, the greatest integer smaller or equal to x.)

(A) $\frac{6}{n!}$ (B) $\frac{27}{n^3}$ (C) $\frac{1}{n}$ (D) $\frac{2}{n}$ (E) $\frac{3}{n}$

13. The picture below shows an equilateral triangle $\triangle SAB$ of side a which is an axial cross section of a right circular cone. Find the radius of the sphere inscribed in the cone.



(A) $\dfrac{a}{2\sqrt{3}}$ (B) $\dfrac{a}{2(1+\sqrt{3})}$ (C) $\dfrac{a}{6}$ (D) $\dfrac{a}{4}$ (E) $\dfrac{a\sqrt{3}}{4}$

14. A class of students took an exam consisting of two problems. 20 students solved problem 1, 19 students solved problem 2, and the number students that did not solve any of the two problems is 10 more than the number of students that solved both problems. Find the number of students in the class.

(A) less than 29 (B) 29 (C) 39 (D) 49 (E) more than 49

15. Assume that $a, x, y > 1$. Which of the following expressions equals $\log_{xy} a$?

(A) $\log_x a + \log_y a$ (B) $\log_x(ay) - \log_y(ax)$ (C) $\dfrac{\log_x a}{1 + \log_x y}$

(D) $\dfrac{\log_y a}{1 - \log_x y}$ (E) $\dfrac{\log_y a}{\log_a x + \log_x a}$

16. Suppose that the function f satisfies the functional identity

$$f(x) + f\left(\min(x+1, 7-2x)\right) = x$$

for every real number x. (The minimum of a and b, $\min(a, b)$, equals a when a is less than or equal to b and equals b when b is less than a.) Find $f(2)$.

(A) 0 (B) 1 (C) 2 (D) -1 (E) -2

17. Consider a regular pentagon with side lengths 1. Connect the midpoints of the sides of the pentagon to form another pentagon. What is the ratio of the area of the smaller pentagon to the area of the larger pentagon?

(A) $\cos^2(3\pi/10)$ (B) $\sin^2(3\pi/10)$ (C) $\frac{1}{2}$
(D) $\sin^2(3\pi/5)$ (E) $\cos^2(3\pi/5)$

18. Suppose that line L_1 and line L_2 are perpedicular. Let m_1 be the slope of L_1 and let m_2 be the slope of line L_2. Suppose that the square of the slope of line L_1 plus eight times the square of the slope of L_2 is 6. For the rational values of m_1 and m_2 compute the quantity $\left(\frac{1}{m_1}\right)^2 + 2m_2^2$.

(A) $\frac{1}{2}$ (B) $\frac{3}{4}$ (C) $\frac{3}{2}$ (D) $\frac{27}{16}$ (E) $\frac{17}{2}$

19. Suppose that x, y, and z are positive real numbers for which $x+y+z=1$ and $\dfrac{z}{x+y} = \dfrac{x}{y+z} = \dfrac{y}{x+z}$. Find xyz.

(A) $1/27$ (B) $\sqrt{2}/27$ (C) $16/81$ (D) $\sqrt{3}/27$ (E) $8/27$

20. A number N is called a *triplet* if it can be written in base b as aaa_b. Find the sum of all triplets for bases b which satisfy $2 \le b \le 5$. Give your answer in base 10.

(A) 356 (B) 383 (C) 425 (D) 482 (E) 512

21. A special type of door lock has a panel with three buttons labeled with the digits 1, 2 and 3. The lock is opened by a sequence of two actions. Each action consists of either pressing one of the buttons or pressing two of them simultaneously. For example (12)(3) is a possible combination. Another possible combination is (1)(2). Note that (12)(3) and (21)(3) are the same combination, since (12) and (21) refer to pressing at the same time the buttons 1 and 2. How many possible lock combinations are there?

(A) 9 (B) 18 (C) 27 (D) 36 (E) 72

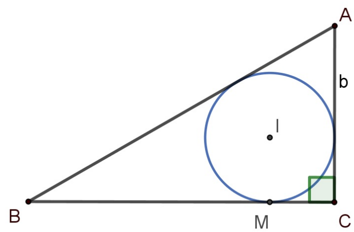
22. Consider the function $f(x) = \frac{1}{x}$. For $a > 0$, denote by L the line with slope $-\frac{1}{a^2}$ that intersects $f(x)$ at the single point $(a, f(a))$. Compute the area of triangle formed by L and the coordinate axes.

(A) 4 (B) 2 (C) 1 (D) $\frac{4}{a}$ (E) $\frac{1}{2a^2}$

23. In a urn there are 5 red balls, 3 white balls, and 2 black balls. One randomly chooses n balls from the urn. Find n such that the probability that among the chosen n balls there is at least one black ball is strictly bigger than $\frac{8}{15}$.

(A) $\{9, 10\}$ (B) $\{8, 9, 10\}$ (C) $\{6, 7, 8, 9, 10\}$ (D) $\{4, 5, 6, 7, 8, 9, 10\}$
(E) any n larger than 2 will do

24. Consider a right triangle $\triangle ABC$, with $\angle C = 90^\circ$ and $\angle A = \frac{\angle B + \angle C}{2}$. Denote $AC = b$ and let M be the tangency point of $\overline{BC}$ with the inscribed circle. Find $BM \cdot MC$.

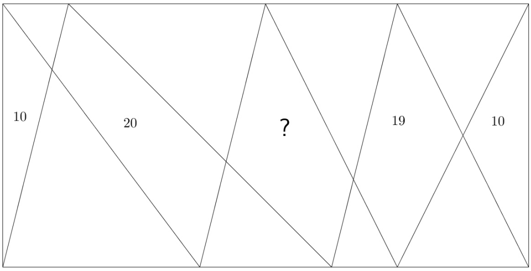


(A) $\frac{b}{\sqrt{3}}$ (B) $\frac{b}{1+\sqrt{3}}$ (C) $\sqrt{3}b^2$ (D) b^2 (E) $\frac{b^2}{2}$

25. The number 14641 has the property that it is a perfect square when interpreted in any base $b \geq 7$. Assuming $b = 7$, find the number x such that $x^2 = 14641$. Here 14641 is written in base 7, but your answer should be given in base 10.

(A) 8 (B) 50 (C) 64 (D) 101 (E) 121

26. A rectangle is divided into various parts by segments with end points on its sides, with areas of four parts marked, as in the graph below.



Find the area of the part with the question mark "?".

(A) $\sqrt{19 \times 20}$ (B) 19 (C) 19.5 (D) 20 (E) 21

27. How many ordered pairs (a, b) of real numbers are there such that $(2a^2+1)+(2a^2-1)\mathbf{i}$ is a solution to the equation $x^2 - 10x + b^2 + 4b + 20 = 0$ (where $\mathbf{i}$ is the imaginary unit)?

(A) 1 (B) 2 (C) 4 (D) 6 (E) 8

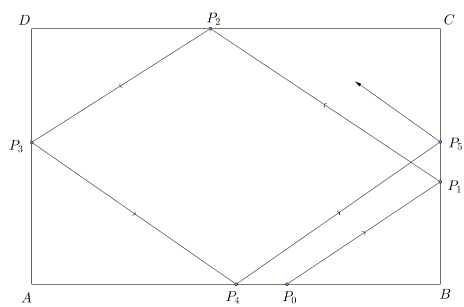
28. Let $\log x$ represent the common logarithmic function. How many ordered 3-tuples (a, b, c) of positive integers are there such that $\log(a+b+c) = \log a + \log b + \log c$?

(A) 0 (B) 1 (C) 6 (D) 24 (E) infinitely many

29. What is the probability that all of the spades are next to each other in an ordinary deck of playing cards?

(A) $\frac{13!}{40!}$ (B) $\frac{13!}{52!}$ (C) $\frac{4 \cdot 13!}{52!}$ (D) $\frac{4 \cdot 40!}{52!}$ (E) $\frac{13! \cdot 40!}{52!}$

30. A ball is moving inside a rectangle $ABCD$ and is bounced/reflected by the sides. It starts from the point P_0 on AB, and reaches side BC at P_1, then reaches side CD at P_2, etc, as illustrated in the picture below. (Suppose that the ball always reaches the interior of the adjacent side for its next bounce.)



The coordinates of A, B, C, D are $(0,0)$, $(18,0)$, $(18,12)$, $(0,12)$, respectively. If $P_0 = (3,0)$ and $P_4 = P_0$, find P_{2019}.

(A) $(0,2)$ (B) $(0,10)$ (C) $(3,12)$ (D) $(15,12)$
(E) different from (A)–(D)

31. Find the area of the region defined by the set of points

$$S = \{(x,y) : 2019 \leq \max(2|x|, 3|y|) \leq 2020\}.$$

(The maximum of a and b, $\max(a,b)$, equals a when a is greater than or equal to b and equals b when b is greater than a.)

(A) $4(3 \cdot 2020 - 2 \cdot 2019)$ (B) $\frac{2}{3}(2020^2 - 2019^2)$ (C) $4(\frac{2020^2}{3} - \frac{2019^2}{2})$
(D) $2 \cdot 2020^2 - 3 \cdot 2019^2$ (E) $\frac{2020 \cdot 2019}{6}$

32. There are four large groups of people, each with 1000 members. Any two of these groups have 100 members in common. Any three have 10 members in common. And there is one person in all four groups. All together, how many people are in these groups?

(A) 3221 (B) 3439 (C) 3617 (D) 3659 (E) 3827

33. We say that x_0 is a *local maximum* of a function $f : \mathbb{R} \to \mathbb{R}$ if there exists an interval $I = (x_0 - a, x_0 + a)$ centered at x_0 such that $f(x_0) \geq f(x)$ for all $x \in I$. Find a condition on the coefficients b and c such that the function $f(x) = |-x^2 + bx + c|$ does not admit a local maximum.

(A) $b = 0$, no condition on c (B) $c = 0$, no condition on b
(C) $b^2 + 4c > 0$ (D) $b^2 + 4c \leq 0$
(E) $f(x)$ has a local maximum for any choice of b and c

34. Two balls are drawn from an urn containing n balls numbered 1 through n. Assume $n > 1$. The first ball is kept if it is numbered 1 and returned otherwise. What is the probability of the second ball being numbered 2?

(A) $\frac{2n-1}{2n(n-1)}$ (B) $\frac{1}{n}$ (C) $\frac{1}{n-1}$ (D) $\frac{n^2-n+1}{n^2(n-1)}$
(E) different from (A) to (D)

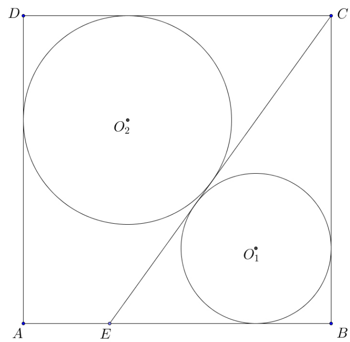
35. If $\cos\theta + \sin\theta = \frac{1}{3}$ for some angle θ, what is the value of $\cos^3\theta + \sin^3\theta$?

(A) $\frac{1}{27}$ (B) $\frac{5}{27}$ (C) $\frac{13}{27}$ (D) $\frac{17}{27}$ (E) different from (A) to (D)

36. 36 houses are arranged in a square shape with 6 rows and 6 columns. They are to be painted either red or blue such that no three consecutive houses in a row are of the same color and no two consecutive houses in a column are of the same color. How many ways are there to paint these houses?

(A) 24^6 (B) 24 (C) 26^6 (D) 26 (E) 28

37. Given a square $ABCD$ with side length 1 and a point E on the side AB, let O_1 be the circle that is tangent to the segments EB, BC, CE, and let O_2 be the circle that is tangent to the segments CD, DA, CE, as illustrated in the picture below.



If the radius of the circle O_1 is $\frac{1}{4}$, find the radius of the circle O_2.

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{2}{5}$ (D) $\frac{1}{4}(7-4\sqrt{2})$ (E) $\frac{3}{4}(\sqrt{2}-1)$

38. Solve the equation

$$3\sin^{-1}x + 2\cos^{-1}x + \tan^{-1}x = \frac{\pi}{2}.$$

(A) $\sqrt{\frac{-1+\sqrt{3}}{4}}$ (B) $-\sqrt{\frac{-1+\sqrt{3}}{4}}$ (C) $-\sqrt{\frac{1+\sqrt{5}}{2}}$
(D) $\sqrt{\frac{-1+\sqrt{5}}{2}}$ (E) $-\sqrt{\frac{-1+\sqrt{5}}{2}}$

39. Consider the equation $2019 = 3^m - p \cdot 3^n$, with variables m, n, and p counting numbers. Find the smallest value of m that appears in a solution (m, n, p) of this equation.

(A) 5 (B) 6 (C) 7 (D) 8 (E) 10

40. The equation $x^3 + px + q = 0$ has real roots x_1, x_2 and x_3 satisfying $\arctan(x_1) + \arctan(x_2) = \arctan(x_3)$. Assuming that $x_3 \neq 0$, find x_3.

(A) $2 - \frac{q^2}{4} - p$ (B) $2\sqrt{2-p}$ (C) $-\frac{q}{1+p}$ (D) $-\frac{p}{2}$ (E) $-\frac{q}{2}$

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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