MMPC STUDENT CODE

SIXTIETH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America

Part II

Wednesday, December 7, 2016

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**

2. Part II consists of problems and proofs. You will be allowed 100 minutes (1 hour and 40 minutes) for the five questions. To receive full credit for a problem, you are expected to justify your answer.

3. You are not expected to solve all problems completely. Look over all the problems and work first on those that interest you the most. If you are unable to solve a particular problem, partial credit might be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.

4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the blank page at the end of the booklet (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page clearly write the **problem number**. If you use additional paper for your answer, check the appropriate box and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.

5. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.

6. The competition rules prohibit you from asking questions of anyone during the examination. The use of notes, reference material, computation aids, or any other aid is likewise prohibited. Please note that **calculators are not allowed** on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. Please do not return scratch paper containing routine numerical calculations.

7. You may now open the test booklet.

#	1	2	3	4	5	Total
Score						

1. If a polygon has both an inscribed circle and a circumscribed circle, then define the *halo* of that polygon to be the region inside the circumcircle but outside the incircle. In particular, all regular polygons and all triangles have halos.
   (a) What is the area of the halo of a square with side length 2?
   (b) What is the area of the halo of a 3-4-5 right triangle?
   (c) What is the area of the halo of a regular 2016-gon with side length 2?

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2. Let $s_1, s_2, s_3, s_4, \ldots$ be a sequence (infinite list) of 1's and 0's. For example $1, 0, 1, 0, 1, 0, \ldots$, that is, $s_n = 1$ if n is odd and $s_n = 0$ if n is even, is such a sequence. Prove that it is possible to delete infinitely many terms in $s_1, s_2, s_3, s_4, \ldots$ so that the resulting sequence is the original sequence. For the given example, one can delete $s_3, s_4, s_7, s_8, s_{11}, s_{12}, \ldots$.

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3. This problem is about pairs of consecutive whole numbers satisfying the property that one of the numbers is a perfect square and the other one is the double of a perfect square.
   (a) The smallest such pairs are $(0,1)$ and $(8,9)$, Indeed $0 = 2 \times 0^2$ and $1 = 1^2$; $8 = 2 \times 2^2$ and $9 = 3^2$. Show that there are infinitely many pairs of the form $(2a^2, b^2)$ where the smaller number is the double of a perfect square satisfying the given property.
   (b) Find a pair of integers satisfying the property that is not in the form given in the first part, that is, find a pair of integers such that the smaller one is a perfect square and the larger one is the double of a perfect square.

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4. It is a fact that every set of 2016 consecutive integers can be partitioned in two sets with the following four properties:
(i) The sets have the same number of elements.
(ii) The sums of the elements of the sets are equal.
(iii) The sums of the squares of the elements of the sets are equal.
(iv) The sums of the cubes of the elements of the sets are equal.

Let $S = \{n+1, n+2, \ldots, n+k\}$ be a set of k consecutive integers.

(a) Determine the smallest value of k such that property (i) holds for S.

(b) Determine the smallest value of k such that properties (i) and (ii) hold for S.

(c) Show that properties (i), (ii) and (iii) hold for S when $k = 8$.

(d) Show that properties (i), (ii), (iii) and (iv) hold for S when $k = 16$.

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5. Consider four real numbers x, y, a, and b, satisfying $x + y = a + b$ and $x^2 + y^2 = a^2 + b^2$. Prove that $x^n + y^n = a^n + b^n$, for all $n \in \mathbb{N}$.

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(Continued Solutions)

The Michigan Mathematics Prize Competition is an activity of the
Michigan Section of the Mathematical Association of America

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