THE SIXTIETH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

Sponsored by

The Michigan Section of the Mathematical Association of America

Part I

Tuesday, October 11, 2016

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Carefully read and follow the instructions printed on the answer sheet. Check to ensure that your six-digit MMPC code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor asks you to stop, please quit working immediately and turn in your answer sheet.

3. Consider the problems and responses carefully. You may work out ideas on scratch paper before selecting a response.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.

5. For each of the questions, five different possible responses are provided. Choose the correct answer and completely fill in the corresponding bubble on your answer sheet.

6. Any scientific or graphing calculator is permitted on Part I. Unacceptable machines include computers, PDAs, pocket organizers, cell phones, and similar devices. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator. No other devices are permitted.

7. No one is permitted to explain to you the meaning of any question. Do not ask anyone to violate the rules of the competition. If you have questions concerning the instructions, ask them now.

8. You may open the test booklet and begin.

1. What is the smallest nonnegative integer k such that the equation $x^2 + kx + 100 = 0$ has two distinct real roots?

   A: 0 B: 3 C: 7 D: 14 E: 21

2. There are 720 integers using each of the digits 1, 2, 3, 4, 5, and 6 exactly once. If we write these integers in increasing order, then 123456 is the first and 654321 is the last. If 615423 is in the nth position, find n.

   A: 622 B: 623 C: 624 D: 625 E: 626

3. Let a and b be real numbers such that $a \neq 1$ and $b \neq 1$. Let $f(x) = ax + b$. Define $f^n(x) = f(f(f(\cdots f(x))\cdots))$ where f appears n times. In particular $f^1(x) = f(x)$, $f^2(x) = f(f(x))$, and $f^3(x) = f(f(f(x)))$. Find $f^n(b)$.

   A: $b(a^{n+1} - 1)/(a - 1)$ B: $b(a^n - 1)/(a - 1)$ C: $a(b^{n+1} - 1)/(b - 1)$
   D: $a(b^n - 1)/(b - 1)$ E: $a^n + b^n$

4. Let a, b, c, and d be positive real numbers. Let L_1, L_2, L_3, and L_4 be four lines in the plane such that their slopes are $1/a$, $1/b$, $1/c$, and $1/d$ respectively. Suppose that the four lines have the same y-intercept, and the sum of the x-intercepts of these four lines is N. Find the common y-intercept.

   A: $N(a + b + c + d)$ B: $-N(a + b + c + d)$ C: $-N/(a + b + c + d)$
   D: $N/(a + b + c + d)$ E: $(a + b + c + d)/N$

5. There are four numbers such that, by selecting them one at a time and adding the average of the remaining three, we generate the four new numbers 1, 11, 111, and 1111. Find the sum of the original four numbers.

   A: 613 B: 614 C: 615 D: 616 E: 617

6. Find the number of integers that can be expressed as the sum of four distinct integers chosen from the set $\{1, 2, 3, \ldots, 2016\}$.

   A: 8049 B: 8050 C: 8051 D: 8052 E: 8053

7. Consider the sequence defined by $a_1 = 1$ and $a_n = a_{n-1} + (-1)^{n+1}n$ for $n \geq 2$. (So $a_1 = 1$, $a_2 = 1 - 2$, $a_3 = 1 - 2 + 3$, $\ldots$.) Find a_{2016}.

   A: 1007 B: -1007 C: 1008 D: -1008 E: 1009

8. A fair die, with 1, 2, 3, 4, 5, and 6 on its six faces, is rolled. If the top face is 1, the process is stopped; otherwise, the number on the top face is replaced by 1 and the die is rolled again. What is the probability that the process stops after exactly three rolls?

   A: 4/18 B: 5/18 C: 6/18 D: 7/18 E: 8/18

9. A cubic box with side length 10 cm is completely filled with water. If this water is poured into another cubic box with side length 20 cm, what is the depth of the water in this box, in cm?

   A: 2.2 B: 2.3 C: 2.4 D: 2.5 E: 2.6

10. An integer is *super even* if all its digits are even. How many 4-digit positive integers x are there such that x, $2x$, and $4x$ are all super even?

   A: 4 B:6 C: 8 D: 10 E: 12

11. Let $N = 2 + 22 + 202 + 2002 + \cdots + 200\cdots02$, where the last summand has 2016 zeros. Find the sum of the digits in N.

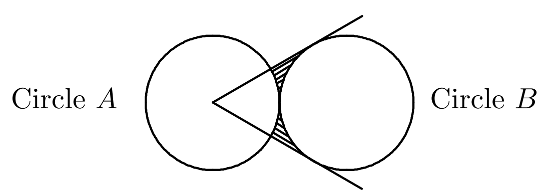
   A: 4041 B: 4043 C: 4045 D: 4047 E: 4049

12. There are 100 students and there are two clubs, the chess club and the glee club. Every student must join at least one club. As a result, at least one student joined the chess club but not the glee club, and at least one student joined the glee club but not the chess club. There are as many chess club members as glee club members. Let n be the maximum number of students that joined both clubs. Find the sum of the digits of n.

   A: 18 B: 17 C: 11 D: 8 E: 7

13. In the figure below, circles A and B are tangent to one another, the two line segments intersect at the center of circle A and are tangent to circle B, and the radius of each circle is 1 unit. Find the area of the shaded region in square units.

   A: $\frac{\pi}{2}$ B: $\sqrt{3}$ C: $\frac{\sqrt{3}}{2} - \frac{\pi}{4}$ D: $\sqrt{3} - \frac{\pi}{4}$ E: $\sqrt{3} - \frac{\pi}{2}$



14. What is the last digit of the number $2012^{2014^{2016}}$?

   A: 0 B: 2 C: 4 D: 6 E: 8

15. A sequence of integers $a_0, a_1, a_2, \ldots, a_n, \ldots$ is formed as follows. The initial term is $a_0 = 0$. If n is odd, then $a_n = a_{n-1} + 2$. If n is even, then $a_n = 3 \cdot a_{n-1}$. Find a_{51}.

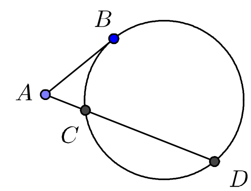
   A: $2 \cdot 26 + 3 \cdot 25$ B: $2 \cdot 26 + 3^{25}$ C: $2^{26} \cdot 3^{25}$ D: $2^{26} \cdot 3 + 2 \cdot 3^{25}$
   E: $3^{26} - 1$

16. Consider a regular heptagon (polygon with seven vertices) inscribed in a circle. How many triangles formed using the vertices of the heptagon contain the center of the circle?

A: 35 B: 21 C: 18 D: 14 E: 10

17. In the picture below, $\overline{AB}$ is tangent to the circle, $AB = 10$, $CD = 15$ and $\overline{BD}$ is a diameter of the circle. Find the length of the segment $\overline{AC}$. (The figure may not be drawn to scale.)

A: 3 B: 4 C: 5 D: 7.5 E: 8



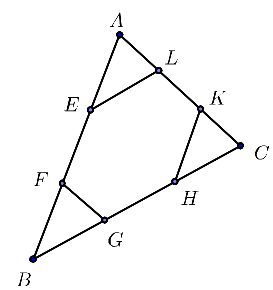
18. Which of the following functions, $g(x)$, gives the best polynomial approximation of $f(x) = 2 - |x - 1|$ over the interval $[1, 3]$?

A: $g(x) = 3 - x$ B: $g(x) = x + 1$ C: $g(x) = 2 - x^2$

D: $g(x) = 1 + \cos\left(\frac{\pi}{2}(x - 1)\right)$ E: $g(x) = 1 + \frac{1}{3}x(x - 2)(x - 4)$

19. In the picture below, E and F trisect $\overline{AB}$, G and H trisect $\overline{BC}$, and K and L trisect $\overline{AC}$. The area of $\triangle ABC$ is 4. Find the area of hexagon $EFGHKL$.

A: 3 B: $\frac{8}{3}$ C: 2.5 D: $\frac{7}{3}$ E: 2



20. How many complex roots does $x^9 - 1 = 0$ have that are not roots of $x^n - 1 = 0$ for any integer n such that $0 < n < 9$?

A: 5 B: 6 C: 7 D: 8 E: 9

21. Let p be a prime number. What is the remainder after division by p of the sum $S = 1 + \sum_{i=1}^{p-1} (p + i)^{p-1}$?

A: 0 B: 1 C: 2 D: $p - 2$ E: $p - 1$

22. Consider an arbitrary point (x, y) on the unit circle in the plane. (This means that x and y satisfy the equation $x^2 + y^2 = 1$.) What interval characterizes the values taken by the expression $x + y$ when (x, y) moves along the entire circle?

   A: $[0, 1]$ B: $[-1, 1]$ C: $[-\sqrt{2}, \sqrt{2}]$ D: $[-\sqrt{3}, \sqrt{3}]$ E: $[-2, 2]$

23. Define an operation $\star$ on $\mathbb{R}\backslash\{0\}$ by the following: $m \star n = m - n + \frac{m}{n}$. (For example, $2 \star 2 = 2 - 2 + \frac{2}{2} = 1$.) Fix $a \in \mathbb{R}\backslash\{-1, 0\}$. Let x_1 be the solution of the equation $x \star a = 1$ and let x_2 be the solution of the equation $x \star a = 0$. What is the value of $x_2 \star x_1$?

   A: $\dfrac{a^2}{a+1}$ B: $\dfrac{1}{a}$ C: a D: 1 E: 0

24. Which of the following numbers is a solution to the equation $586_x = 2016_7$? (Here, the index represents the base in which the number is written. For example, 102_3 is the number in base 3 that equals 11 in ordinary base 10.)

   A: 7 B: 9 C: 11 D: 13 E: 15

25. A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find the probability of obtaining a number strictly less than 4 on a single roll of the die.

   A: $\dfrac{4}{9}$ B: $\dfrac{1}{2}$ C: $\dfrac{5}{9}$ D: $\dfrac{2}{3}$ E: none of the others

26. Four candidates are seeking a vacancy on a school board. Candidate A is twice as likely to be elected as candidate B, and B and C have the same chance of being elected, while candidate C is twice as likely to be elected as candidate D. What is the probability that A will not win?

   A: $\dfrac{1}{4}$ B: $\dfrac{4}{9}$ C: $\dfrac{1}{2}$ D: $\dfrac{5}{9}$ E: $\dfrac{7}{8}$

27. Which of the following best describes the region in the plane defined by the inequality $|x + 1| + |y - 1| < 2$?

   A: a disk B: a square C: a region bounded by two parabolas
   D: a cardioid E: a hypocycloid

28. It is a fact that any rhombus has an inscribed circle. Consider a rhombus with length of sides 5 and area 5. What is the radius of its inscribed circle?

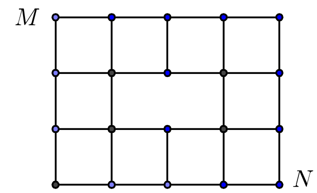
   A: $\dfrac{2}{5}$ B: $\dfrac{1}{2}$ C: 1 D: $\dfrac{5}{4}$ E: $\dfrac{5}{2}$

29. In which of the following bases will the expansion of 2016! end with the most zeros?

   A: 7 B: 8 C: 9 D: 10 E: 11

30. How many paths from M to N are possible in the grid below, knowing that one can only move down or to the right in the grid?

A: 10 B: 15 C: 18 D: 26 E: 35



31. Let $f(x) = -x^2 + 6x - 8$. Determine the sum of the x-intercepts and the y-intercept of $f(x-3)$.

A: 12 B: 0 C: -12 D: -20 E: -23

32. A line segment connects the point $(0,0)$ to $(20,16)$. Determine the number of lattice points that the line segment crosses (not including $(0,0)$ and $(20,16)$).

A: 2 B: 3 C: 4 D: 5 E: 6

33. Determine the area of the region in the xy plane defined by the intersection of $a \leq |x-y| \leq b$ and $a \leq |x+y| \leq b$, where $0 < a < b$.

A: $2ab$ B: $4ab$ C: $(b-a)^2$ D: $2(b-a)^2$ E: $4(b-a)^2$

34. There is a set of three six-sided dice. Die A has sides of 2, 2, 4, 4, 9, and 9. Die B has sides of 1, 1, 6, 6, 8, and 8. Die C has sides of 3, 3, 5, 5, 7, and 7. Two dice are rolled and whichever die has the higher number showing wins. Calculate the following sum of probabilities: (probability that Die A wins against die B) + (probability that Die B wins against Die C) + (probability that Die C wins against Die A).

A: 2 B: 3/2 C: 5/3 D: 8/5 E: 13/8

35. Let $S = \{3, 4, 5, 6\}$. Using each element of S exactly once, create a pair of two-digit numbers (for example 34 and 56, or 36 and 54). Find the difference of the maximum product and the minimum product of these numbers.

A: 1760 B: 1772 C: 1782 D: 1792 E: 1804

36. Let a and b represent the lengths of the legs of a right triangle and let c represent the length of the hypotenuse of the triangle. Which of the following expressions is numerically equivalent to the area of the triangle?

A: $e^{\ln \frac{a}{\sqrt{2}} + \ln \frac{b}{\sqrt{2}}}$ B: $\frac{1}{2}bc\sin a$ C: $(a+b)^2 - (a^2+b^2)$

D: $a^2 + b^2 - c^2$ E: $\ln\left(e^{\frac{a}{2}}e^{\frac{b}{2}}\right)$

37. A solid is generated by taking the triangle in the xy-plane formed by connecting the points $(4,0)$, $(4,2)$, and $(6,0)$ and revolving the region around the y-axis. What is the volume of the resulting solid swept out by the revolution?

A: $\dfrac{8\pi}{3}$ B: $\dfrac{56\pi}{3}$ C: $\dfrac{64\pi}{3}$ D: $\dfrac{100\pi}{3}$ E: $\dfrac{112\pi}{3}$

38. Katrina and Leslie are running a race that will consist of a sprint followed by hurdles. Katrina can sprint at 12 miles per hour, but her speed falls to 8 miles per hour during hurdles. Leslie sprints at 14 miles per hour, but her speed drops to 7 miles per hour for hurdles. If the sprint has a distance of 1 mile, how long does the hurdles portion of the race have to be in order to guarantee that Katrina wins?

A: More than $\frac{3}{4}$ of a mile B: More than 1 mile

C: More than $\frac{2}{3}$ of a mile D: More than $\frac{1}{2}$ of a mile

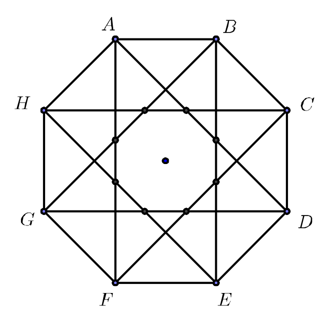
E: More than $\frac{3}{5}$ of a mile

39. Mildred drives at a rate of 20 miles per hour to visit her friend Ethel. When driving the same distance home, how fast will she have to drive to average a rate of 30 miles per hour for the whole trip?

A: 40 mph B: 45 mph C: 50 mph D: 55 mph E: 60 mph

40. In the regular octagon $ABCDEFGH$, every third vertex is connected (A to D, B to E, etc.) to form a smaller octagon with the same center, as seen in the figure below. Determine the ratio of the area of the smaller octagon to the area of the original octagon.

A: $3-2\sqrt{2}$ B: $\sqrt{2}-1$ C: $\sqrt{2}/4$ D: $1/8$ E: $3\sqrt{2}-4$



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ACKNOWLEDGMENTS

- The Michigan Association of Secondary School Principals has placed this competition on the Approved List of Michigan Contests and Activities.
- We wish to thank Eastern Michigan University for supporting this competition.
- We wish to thank Albion College for hosting the MMPC grading day.

(1)	E	(11)	D	(21)	A	(31)	E
(2)	B	(12)	B	(22)	C	(32)	B
(3)	A	(13)	E	(23)	E	(33)	D
(4)	C	(14)	D	(24)	C	(34)	C
(5)	E	(15)	E	(25)	C	(35)	D
(6)	A	(16)	D	(26)	D	(36)	A
(7)	D	(17)	C	(27)	B	(37)	B
(8)	B	(18)	A	(28)	B	(38)	C
(9)	D	(19)	B	(29)	B	(39)	E
(10)	C	(20)	B	(30)	D	(40)	A