MMPC STUDENT CODE

FIFTY-NINTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America

Part II

Wednesday, December 9, 2015

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**
2. Part II consists of problems and proofs. You will be allowed 100 minutes (1 hour and 40 minutes) for the five questions. To receive full credit for a problem, you are expected to justify your answer.
3. You are not expected to solve all problems completely. Look over all the problems and work first on those that interest you the most. If you are unable to solve a particular problem, partial credit might be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.
4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the blank page at the end of the booklet (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page clearly write the **problem number**. If you use additional paper for your answer, check the appropriate box and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.
5. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.
6. The competition rules prohibit you from asking questions of anyone during the examination. The use of notes, reference material, computation aids, or any other aid is likewise prohibited. Please note that **calculators are not allowed** on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. Please do not return scratch paper containing routine numerical calculations.
7. You may now open the test booklet.

#	1	2	3	4	5	Total
Score						

1. Consider a right triangle with legs of lengths a and b and hypotenuse of length c such that the perimeter of the right triangle is numerically (ignoring units) equal to its area. Prove that there is only one possible value of $a + b - c$, and determine that value.

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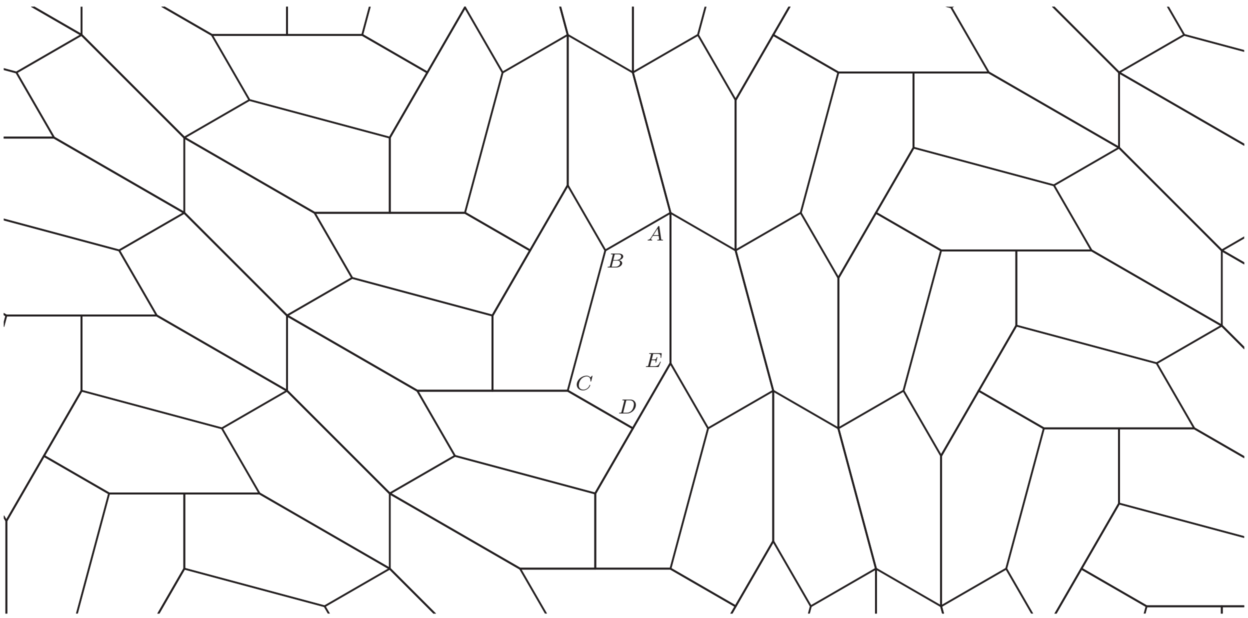
2. Last August, Jennifer McLoud-Mann, along with her husband Casey Mann and an undergraduate David Von Derau at the University of Washington, Bothell, discovered a new tiling pattern of the plane with a pentagon. This is the fifteenth pattern of using a pentagon to cover the plane with no gaps or overlaps. It is unknown whether other pentagons tile the plane, or even if the number of patterns is finite. Below is a portion of this new tiling pattern.



Determine the five angles (in degrees) of the pentagon $ABCDE$ used in this tiling. Explain your reasoning, and give the values you determine for the angles at the bottom.

$A =$

$B =$

$C =$

$D =$

$E =$

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3. Let $f(x) = \sqrt{2019 + 4\sqrt{2015}} + \sqrt{2015}\,x$. Find all rational numbers x such that $f(x)$ is a rational number.

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4. Alice has a whiteboard and a blackboard. The whiteboard has two positive integers on it, and the blackboard is initially blank. Alice repeats the following process.

   Let the numbers on the whiteboard be a and b, with $a \leq b$.
   Write a^2 on the blackboard.
   Erase b from the whiteboard and replace it with $b - a$.

   For example, if the whiteboard began with 5 and 8, Alice first writes 25 on the blackboard and changes the whiteboard to 5 and 3. Her next move is to write 9 on the blackboard and change the whiteboard to 2 and 3.

   Alice stops when one of the numbers on the whiteboard is 0. At this point the sum of the numbers on the blackboard is 2015.

a. If one of the starting numbers is 1, what is the other?

b. What are all possible starting pairs of numbers?

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5. Professor Beatrix Quirky has many multi-volume sets of books on her shelves. When she places a numbered set of n books on her shelves, she doesn't necessarily place them in order with book 1 on the left and book n on the right. Any volume can be placed at the far left. The only rule is that, except the leftmost volume, each volume must have a volume somewhere to its left numbered either one more or one less. For example, with a series of six volumes, Professor Quirky could place them in the order 123456, or 324561, or 564321, but not 321564 (because neither 4 nor 6 is to the left of 5).

   Let's call a sequence of numbers a *quirky sequence of length n* if:

   1. the sequence contains each of the numbers from 1 to n, once each, and
   2. if k is not the first term of the sequence, then either $k+1$ or $k-1$ occurs somewhere before k in the sequence.

   Let q_n be the number of quirky sequences of length n. For example, $q_3 = 4$ since the quirky sequences of length 3 are 123, 213, 231, and 321.

a. List all quirky sequences of length 4.

b. Find an explicit formula for q_n. Prove that your formula is correct.

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(Continued Solutions)

The Michigan Mathematics Prize Competition is an activity of the
Michigan Section of the Mathematical Association of America

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