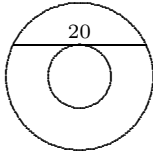# FIFTY-NINTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION
Sponsored by
The Michigan Section of the Mathematical Association of America
Part I
Tuesday, October 6, 2015
INSTRUCTIONS
(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Carefully read and follow the instructions printed on the answer sheet. Check to ensure that your six-digit MMPC code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor asks you to stop, please quit working immediately and turn in your answer sheet.

3. Consider the problems and responses carefully. You may work out ideas on scratch paper before selecting a response.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.

5. For each of the questions, five different possible responses are provided. Choose the correct answer and completely fill in the corresponding bubble on your answer sheet.

6. Any scientific or graphing calculator is permitted on Part I. Unacceptable machines include computers, PDAs, pocket organizers, cell phones, and similar devices. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator. No other devices are permitted.

7. No one is permitted to explain to you the meaning of any question. Do not ask anyone to violate the rules of the competition. If you have questions concerning the instructions, ask them now.

8. You may open the test booklet and begin.

1. How many ways are there to give 25 cents in change using dimes, nickels, and pennies?

   a) 4   b) 8   c) 10   d) 11   e) 12

2. An 8.5 inch by 11 inch sheet of paper has squares of side length x inches cut out of each corner. The flaps are folded up as the sides of an open-top box. What is the volume of this box (in cubic inches) in terms of x?

   a) $93.5x$ b) $93.5 - 4x^2$ c) $x(8.5 - x)(11 - x)$
   d) $x(8.5 - 2x)(11 - 2x)$ e) $2x(8.5 - 2x)(11 - 2x)$

3. Let $f(x) = a \cdot b^x$ with $b > 0$. If $f(2) = 8$ and $f(6) = 2048$, what is $a + b$?

   a) 4.5   b) 5   c) 6   d) 8   e) 8.5

4. You purchase a $100 item with successive discounts of 20%, 10%, 5%, and 5%. Your friend purchases the item with successive discounts of 10%, 10%, 10%, and 10%. The second, third, and fourth discounts are applied to the discounted price, not to the original price. The lower of these prices is less than the higher by how much?

   a) Nothing. You both paid the same price.
   b) A fraction of a cent.   c) 63 cents.   d) 88 cents.
   e) It depends on the order in which the discounts are applied.

5. What is the last (units) digit when 3^{2015} is written in standard decimal notation?

   a) 1   b) 3   c) 5   d) 7   e) 9

6. The complex number z has imaginary part equal to 6 and satisfies $\dfrac{z}{z+k} = 2i$ for a real number k. Determine the value of k.

   a) -15 b) -12 c) -9 d) 9   e) 15

7. If you reverse the standard order of operations, what is the value of $3 + 5 \wedge 2 \times 4$? Note: The symbol $\wedge$ means "raised to the power of".

   a) 103   b) 256   c) 614656   d) 390628   e) 16777216

8. If A, B, C are real numbers such that
$$x^2 = A(x-1)(x-2) + B(x-1)(x-3) + C(x-2)(x-3)$$
for all real numbers x, what is $A + C$?

a) 0   b) 1   c) 2.5   d) 5   e) 10

9. Define $g(n) = 2^n$. If $g(g(g(3)))g(g(3))g(3) = g(k)$, then what is the sum of the digits of k?

a) 7   b) 9   c) 11   d) 13   e) 15

10. Virginia spends the day painting. It takes 35 minutes to complete each painting, and she takes a 20 minute break between each painting and the next. If Virginia starts her third painting at 1:00 pm, at what time does she finish her seventh painting?

a) 3:55 pm   b) 4:40 pm   c) 4:55 pm   d) 5:15 pm   e) 5:35 pm

11. Consider the two concentric circles in the diagram. The outer circle has a chord of length 20 that is tangent to the inner circle. What is the area of the annular region between the two circles?

a) 10π b) 100π c) 200π d) 400π
e) The area cannot be determined from the given information.

12. My car displays my fuel efficiency in a bar graph. At the end of each minute, a new bar is added to the graph with the miles per gallon (mpg) for that minute. I get 42 mpg, 48 mpg, and 40 mpg in the first three minutes of a trip. What fuel consumption must I achieve in the next minute to have an average of 50 mpg for the first four minutes?

a) 50 mpg   b) 70 mpg   c) 80 mpg
d) It is impossible to achieve an average of 50 mpg for the first four minutes.
e) The fuel consumption to achieve an average of 50 mpg cannot be determined from the given information.

13. How many digits are required to write 2015^{2015} as an integer in standard decimal notation?

a) 7   b) 6658   c) 6659   d) 15330   e) 15331

14. Suppose P and Q are statements (sentences that are either true or false). For what truth values of P and Q will the following statement be false:

$$(P \text{ or } Q) \text{ implies } P$$

a) Both P and Q are true.
b) P is true and Q is false.
c) P is false and Q is true.
d) Both P and Q are false.
e) No truth values of P and Q make the statement false.

15. What is the remainder when $x^{2015} + 2015$ is divided by $x + 1$?

a) 0   b) 1   c) 2014   d) 2015   e) 2016

16. If $-2 \le x \le 5$, the expression $\left| |x + 2| - 7 \right|$ simplifies to which of the following?

a) $5 - x$ b) $x - 5$ c) $9 - x$ d) $x - 9$ e) $x + 9$

17. How many values of θ with $0 \le \theta \le \pi$ satisfy $\sin 2\theta = \cos \theta$?

a) 0   b) 1   c) 2   d) 3   e) an infinite number

18. Find the area of the triangle connecting the points $(2015, 0)$, $(0, 2015)$, and $(20, 15)$ in the Cartesian plane.

a) 1994250   b) 1994550   c) 1994850   d) 1995150   e) 2030150

19. The numbers 2013, 2014, and 2015 can all be expressed as the product of three distinct primes. Compute the sum of those primes.

a) 192   b) 194   c) 196   d) 198   e) 200

20. Bob, Cap, Dave, and Eddie live in apartments A, B, C, and D. Each apartment has exactly one tenant, and the tenant's name does not contain the apartment's letter. How many living configurations are possible?

a) 2   b) 4   c) 6   d) 8   e) 10

21. If the side length of a cube is increased by 2 inches, its volume increases by 386 cubic inches. By how many cubic inches will the volume increase if the side length of the original cube is increased by 3 inches?

    a) 513   b) 657   c) 819   d) 999   e) 1197

22. Suppose we list all the positive integers that are multiples of 3 or 5. This list begins $3, 5, 6, 9, 10, 12, 15, 18, 20, \ldots$. What is the 50th number in the list?

    a) 102   b) 105   c) 108   d) 110   e) 111

23. At a bank, a personal identification number (PIN) is created by selecting an ordered sequence of 4 digits. Define a PIN to be a "pin-up" if the digits of the PIN form a strictly increasing sequence (so 0567 and 1248 are pin-ups, but 1124 and 1729 are not pin-ups). How many PINs are pin-ups?

    a) 126   b) 210   c) 252   d) 3024   e) 5040

24. Let r and s denote the roots of $3x^2 - 13x + 12 = 0$. Determine the value of $\log_2 r + \log_2 s$.

    a) 0   b) 1/2   c) 1   d) 2   e) 3

25. For the sake of simplicity, assume that the orbits of Venus and the earth are circles with the sun at the center, that the circles are in the same plane, that the radii are 108,200,000 km for Venus and 149,600,000 km for the earth. Under these simplifying asumptions, what is the maximum possible angle (to the nearest degree) between Venus and the sun as viewed from the earth?

    a) 36°   b) 44°   c) 46°   d) 54°   e) 63°

26. A geneticist has 100 seeds with one or more of the following traits: wrinkled, yellow, spotted. Among these, 60 are wrinkled, 50 are yellow, 30 are spotted, and 20 have exactly two of these traits. How many of these seeds have all three traits?

    a) 0   b) 10   c) 20
    d) It is impossible to have the traits occur the given number
        of times.
    e) It is impossible to determine from the given information.

27. Let $ABCD$ be a rectangle in which the length of AB is twice the length of BC. Suppose A and B are on the curve $y = 1 - 2x^2$, C and D are on the x-axis, and the rectangle is below the x-axis. Find the perimeter of the rectangle.

    a) 2   b) 3   c) 4   d) 5   e) 6

28. An urn has 10 red balls, 10 blue balls, and n green ball(s). Three balls are selected from the urn at random (without replacement). The probability that the three balls have distinct colors is $10/133$. Find n.

    a) 1   b) 2   c) 3   d) 4   e) 5

29. Suppose $x^2 + bx + c = 0$, where b and c are positive, has two distinct real solutions s and t. What are the solutions to $x^2 + (2c - b^2)x + c^2 = 0$?

    a) s^2 and t^2 b) $-s^2$ and t^2 c) s^2 and $-t^2$
    d) $-s^2$ and $-t^2$ e) st being repeated

30. Suppose $\cos x = 2015 \sin x$. Find $\sin(2x)$.

    a) $4030/(4030^2 + 1)$ b) $4030/(2015^2 - 1)$
    c) $-4030/(2015^2 - 1)$ d) $2015/(2015^2 + 1)$
    e) $4030/(2015^2 + 1)$

31. If $x^2 + 1/x^2 = 2015$ and $x - 1/x$ is positive, what is $x - 1/x$?

    a) $\sqrt{2013}$ b) $\sqrt{2014}$ c) $\sqrt{2015}$ d) $\sqrt{2016}$ e) $\sqrt{2017}$

32. Let p be a prime number. Suppose the sum of p consecutive positive integers is 2015. How many possible values of p are there?

    a) 3   b) 4   c) 5   d) 6   e) 7

33. How many ordered pairs of real numbers (a, b) have the property that $a + b$, $a - b$, ab, a/b are all defined and three of them have the same value?

    a) 0   b) 1   c) 2   d) 3   e) 4

34. An ordinary six-sided die is rolled repeatedly, stopping the first time the 6 appears. What is the probability that each of the numbers is rolled at least once?

a) 1/32   b) 1/10   c) 1/6   d) 2/5   e) 5/6

35. Let $A = \{-1, 0, 3, 8, \dots\}$ be the set of integers that are 1 less than a perfect square, and let $B = \{1000, 1001, 1004, 1009, \dots\}$ be the set of integers that are 1000 more than a perfect square. How many integers are in the intersection $A \cap B$?

a) 4   b) 7   c) 8   d) 15   e) 16

36. A mouse is located at the point $(0, 0)$. Each time the mouse moves from a point (a, b), it goes at random to one of the three equally likely destinations $(a + 1, b + 1)$, $(a + 1, b - 1)$, or $(a - 2, b)$. What is the probability that, after moving six times, the mouse will be at $(0, 0)$ again?

a) 0   b) 1/90   c) 4/81   d) 10/81   e) 2/9

37. If the sum $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}$ is written as a single fraction $\dfrac{A}{9!}$, then what is the remainder when A is divided by 5?

a) 0   b) 1   c) 2   d) 3   e) 4

38. The heptagon $ABCDEFG$ satisfies $AB = BC = DE = FG$ and $CD = EF = GA = 2$, and the vertices all lie on a circle of radius 2. What is the length of AC?

a) $\sqrt{5}$ b) $1 + \sqrt{3}$ c) $2\sqrt{2}$ d) $\sqrt{2} + \sqrt{3}$ e) $2 + \sqrt{2}$

39. When written in base ten, 9^{2000} is 3 followed by 1908 digits. How many of the numbers in the set $\{1, 9, 81, 729, \dots, 9^k, \dots, 9^{2000}\}$ have 9 as the leading digit?

a) 57   b) 92   c) 131   d) 200   e) 222

40. The graph of $2y^3 + xy^2 + 3x^2 = x^3 + 2x^2y + 3y^2$ divides the xy-plane into how many regions?

a) 4   b) 5   c) 6   d) 7   e) 8

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Answers to the

59th Michigan Mathematics Prize Competition

Part I Exam

1 e	9 e	17 d	25 c	33 c
2 d	10 d	18 c	26 b	34 c
3 a	11 b	19 d	27 e	35 a
4 c	12 e	20 b	28 a	36 d
5 d	13 c	21 b	29 a	37 b
6 e	14 c	22 c	30 e	38 c
7 e	15 c	23 b	31 a	39 b
8 d	16 a	24 d	32 b	40 d