FIFTY-EIGHTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

Sponsored by

The Michigan Section of the Mathematical Association of America

Part I

Tuesday, October 7, 2014

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Carefully read and follow the instructions printed on the answer sheet. Check to ensure that your six-digit MMPC code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.
2. Do as many problems as you can in the 100 minutes allowed. When the proctor asks you to stop, please quit working immediately and turn in your answer sheet.
3. Consider the problems and responses carefully. You may work out ideas on scratch paper before selecting a response.
4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.
5. For each of the questions, five different possible responses are provided. In some cases the fifth alternative is "(E) none of the others." If you believe none of the first four alternatives is correct, choose response (E).
6. Any scientific or graphing calculator is permitted on Part I. Unacceptable machines include computers, PDAs, pocket organizers, cell phones, and similar devices. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator. No other devices are permitted.
7. No one is permitted to explain to you the meaning of any question. Do not ask anyone to violate the rules of the competition. If you have questions concerning the instructions, ask them now.
8. You may open the test booklet and begin.

1. Start with a right triangle. Let a and b be the areas of the two semicircles drawn on the two legs of this triangle as diameters. Let c be the area of the semicircle drawn on the hypotenuse of this triangle as diameter. Which relationship regarding a, b and c is correct?
A $a + b = c$ B $a + b = 2c$ C $a^2 + b^2 = c^2$ D $a^2 + b^2 = 2c^2$
E No relationship can be deduced.

2. A rectangular field is three times as long as it is wide. It is enclosed by p meters of fencing. The area of the field in terms of p is
A $\frac{p^2}{64}$ B $\frac{3p^2}{64}$ C $\frac{p^2}{16}$ D $\frac{3p^2}{16}$ E none of the others

3. The batting average of a baseball player is the quotient (number of hits)/(number of at-bats). Each time a player has an "at-bat" the player will either have a hit or not have a hit. A player goes into a game with a .302 batting average in 500 at-bats. At the end of the game, after 5 at-bats, the batter's average is .307. How many hits did the batter have during the game?
A 1 B 2 C 3 D 4 E 5

4. The angle between the hands of a clock at 1:25 is
A $105°$ B $107.5°$ C $110°$ D $112.5°$ E $120°$

5. How many different combinations of six coins, all of them nickels, dimes, and quarters, have a total value of 90 cents?
A 0 B 1 C 2 D 3 E none of the others

6. Find the area of the region consisting of points in the xy-plane satisfying $2013 \leq |x| + |y| \leq 2014$.
A 4027 B 8054 C 16108 D 32216 E none of the others

7. You have available plenty of square tiles of sizes 1×1, 2×2, 3×3, and 4×4. What is the smallest number of such tiles required to cover a 5×5 square with no overlaps and no gaps?
A 6 B 8 C 10 D 13 E none of the others

8. Let p and q represent statements that are either true or false. Let $p \rightarrow q$ denote the statement "p implies q", that is, "If p then q". Under what conditions on the truth values of p and q will the compound statement "$(p \rightarrow q)$ and $(q \rightarrow p)$" be true?
A The truth values of p and q must be the same (both true or both false).
B The truth values of p and q must be different (one true and the other false).
C p must be true and q must be true.
D Any truth values of p and q will make the statement true.
E No truth values of p and q will make the statement true.

9. How many real solutions are there for the equation $|x-1|+|x+1|=2$?
A none B 1 C 3 D between 100 and 200
E infinitely many

10. How many pairs of integer solutions (x, y) are there to $\frac{25^x}{5^{2x+y}} = 5$?
A 0 B 1 C 5 D an infinite number E none of the others

11. If C is the curve with equation $y = ax^3 + bx^2 + cx + d$, $a \neq 0$, and L is the line with equation $y = Ax + B$, $A \neq 0$, which of the following statements could possibly be true of the set S consisting of the points common to both C and L?
A S is empty. B S consists of a single point. C S contains 4 points.
D S contains 5 points.
E More than one of the other statements could be true.

12. The repeating decimal $3.151515\cdots$ is expressed as a rational number in lowest terms. The sum of the numerator plus the denominator is
A 1315 B 415 C 411 D 137 E 38

13. What is the largest integer n such that 2014! is divisible by 11^n?
A 200 B 199 C 184 D 183 E none of the others

14. The measures of the angles of a triangle form an arithmetic progression. Which of the following is true?
A One of the angles must be $60°$.
B None of the angles can exceed $90°$.
C The triangle must be an equilateral triangle.
D The measures of the angles expressed in degrees must all be integers.
E No such triangle exists.

15. A, B, and D are points on the circumference of a circle of radius 1. C is the center of the circle. Angle BAD is 45°. What is the distance from B to D?
A 1 B $\sqrt{2}$ C 2
D It cannot be determined without further information.
E none of the others

16. You have 2014 pennies, all heads up. You choose $20 \times 14 = 280$ coins and turn them over (from heads to tails). You then choose 20 coins and turn these over (from heads to tails or tails to heads, depending on whether each was previously flipped). Then, you choose 14 coins and turn them over. Finally, you count the number of heads showing. How many different answers might you get?
A 34 B 35 C 68 D 69 E 70

17. I write each of the ten numbers $1, 2, 3, 4, \ldots, 10$ on a separate slip of paper and put the slips into a hat. I then write each of the ten numbers $11, 12, 13, \ldots, 20$ on a separate slip of paper and put these slips into a second hat. Someone randomly selects a slip from each hat, so that each slip is equally-likely to be picked. What is the probability that the product of the two numbers will be a multiple of 3?
A 50% B 51% C 60% D $66\frac{2}{3}\%$ E none of the others

18. How many possible choices are there for a (real numbers) if $x^2 - 2x = a^3 + a - 1$ has exactly one solution?
A 0 B 1 C 2 D 3 E infinitely many

19. Three numbers a, b and c are picked at random from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ with replacement. What is the probability that $ab + 2c$ is even?
A $1/4$ B $1/2$ C $46/81$ D $56/81$ E none of the others

20. Twelve points on a circle are used as the vertices of polygons. Which of the following is largest?
A The number of triangles with these vertices.
B The number of quadrilaterals with these vertices.
C The number of pentagons with these vertices.
D The number of hexagons with these vertices.
E The number of heptagons (7-sided polygons) with these vertices.

21. There are 100 seniors in the high school. 36 of them play a musical instrument. 33 speak a second language; and 35 are athletes involved in sports. 34 of the students are involved in none of those activities.
Some of them are involved in more than one activity. In fact, 15 are involved in sports, play musical instruments, and speak a second language. If 20 of the athletes neither play a musical instrument nor speak a second language, how many students both speak a second language and play a musical instrument but are not involved in sports?
A 8 B 23 C 25 D 28
E It cannot be determined without further information.

22. Datura has a collection of different rings. Each day she chooses one ring for her left hand and another for her right hand. If she had four more rings, she would have 100 more ways to make this choice than she has now. How many rings does Datura have now?
A 10 B 11 C 15 D 26
E The problem, as stated, has no solution.

23. Suppose the union of two sets A and B satisfies $A \cup B = \{1, 2, 3\}$. How many possible ordered pairs (A, B) are there?
A 8 B 9 C 15 D 27 E 54

24. Five distinct points are chosen in three-dimensional space. There are ten line segments joining pairs of these five points. What is this maximum possible number of these segments of length 1?
A 6 B 7 C 8 D 9 E 10

25. Consider the product of all the positive integer divisors of 10000. How many digits does this number have?
A 101 B 91 C 51 D 50 E 5

26. A group of children has a collection of N marbles, where N is a three-digit number. When two of the children try to divide up the marbles evenly, there is one marble left over. When three of the children try to divide up the marbles evenly, there is one marble missing. When five of the children try to divide up the marbles evenly, they can do so with no remainder. How many different values of N are consistent with this information?
A none B 10 C 30 D 60 E 100

27. A googol is 10^{100}. A googolplex is 10^{googol}.
The number $x = \log_2(\text{googolplex})$ satisfies
A $x \leq 10^{99}$ B $10^{99} < x \leq 5 \cdot 10^{99}$
C $5 \cdot 10^{99} < x \leq 10^{100}$ D $10^{100} < x \leq 5 \cdot 10^{100}$ E $5 \cdot 10^{100} < x$

28. Consider a regular pentagon with side length 2 units. What is the area of the region consisting of all points that are outside the pentagon but no more than 1 unit distance from it?
A $10 + \pi$ B $10 + 4\pi$ C $20 + 2\pi$ D $4\sqrt{5} + 4\pi$
E none of the others

29. If $g(n)$ is the sum of the cubes of the digits of the positive integer n, then which of the following is closest to the largest possible value of $g(n) - n$?
A 800 B 1000 C 1200 D 1400 E 1600

30. The circle O is tangent to all four sides of quadrilateral $ABCD$. If $AB = 15$, $BC = 12$, $CD = 6$, what is the length AD?
A 6 B 7.5 C 9 D 10.5 E none of the others

31. The squares in a 5×5 grid are labelled with the numbers $-12, -11, -10, \ldots, 11, 12$ in such a way that each number is used exactly once, and for each positive integer $1 \leq n \leq 12$, the numbers n and $-n$ are in squares that share an edge. Among all possible such labellings, how many different squares might contain 0?
A 9 B 12 C 13 D 16 E 25

32. An equilateral triangle, 10 inches on a side, is divided into 100 smaller triangles, each 1 inch on a side. Each of the small triangles will have 0, 1, 2, or 3 corners lying on a side of the big triangle. How many will have exactly one corner on a side of the original triangle?
A 30 B 27 C 24 D 21 E 18

33. A permutation $a_1, a_2, \dots, a_{99}$ of $1, 2, 3, \dots, 99$ is *fair* if $(a_1 - 1^2)(a_2 - 2^2) \cdots (a_{99} - 99^2)$ is even; and *foul* otherwise. How many permutations are fair?
A 1 B $99!/2$ C $99! - 1$ D 99!
E none of the others

34. Which of the following integers cannot be written as a sum of two perfect squares?
A 193 B 197 C 199 D 200 E none of the others

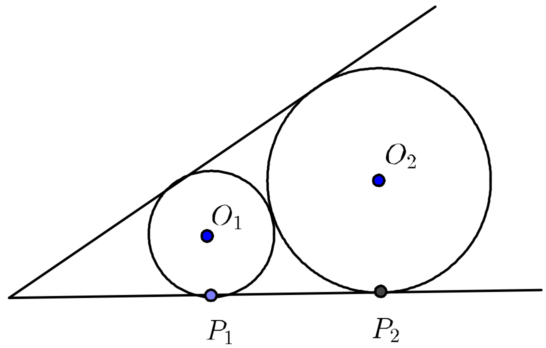
35. Suppose a function f satisfies $f(a) + 100f(2014 - a) = a$ for every even integer a. Find $f(10)$.
A $9990/200390$ B $9999/200390$ C $200390/9999$
D $200390/9990$ E none of the others

36. How many solutions are there to the equation $\tan x + 1 = \frac{1}{2}\cos x$ where $0 \le x \le 2\pi$?
A 0 B 1 C 2 D 3 E none of the others

37. Define $f(n)$ to be the sum of 10^k, where k runs over the digits of n (written in base ten). For example, $f(2014) = 10^2 + 10^0 + 10^1 + 10^4 = 10111$. For how many integers $n < 10{,}000{,}000$ is it true that $f(n) = 2014$?
A 30 B 45 C 60 D 105 E none of the others

38. Let $x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$. Then
A x is an integer.
B x is a rational number, but not an integer.
C x is an irrational number.
D there is more than one possible value for x.
E the expression diverges to infinity.

39. Two circles O_1, O_2 are tangent to each other and to both legs of an angle. The points of tangency with one of the legs are at P_1, P_2. If the radii of the circles are 9 and 16, then how long is segment P_1P_2?



A 12 B $2\sqrt{7}+10$ C 24 D 25 E $12\sqrt{7}$

40. Gabriel and Charles are playing a game of chance. They have a bag with 26 tiles, each containing one of the letters of the English alphabet, each letter appearing on exactly one tile. They form a sequence of letters by drawing tiles from the bag randomly, one at a time, without replacement. The winner is the first person whose full first name can be spelled using the letters drawn. If both achieve this simultaneously, then the game is a tie. What is the probability of a tie?
A 2/13 B 1/3 C 2/5 D 4/7 E none of the others

Answer Key

1 A	11 B	21 A	31 C
2 B	12 D	22 B	32 D
3 D	13 A	23 D	33 D
4 B	14 A	24 D	24 C
5 C	15 B	25 C	35 C
6 B	16 B	26 C	36 C
7 B	17 B	27 D	37 B
8 A	18 B	28 A	38 A
9 E	19 D	29 D	39 C
10 D	20 D	30 C	40 C

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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