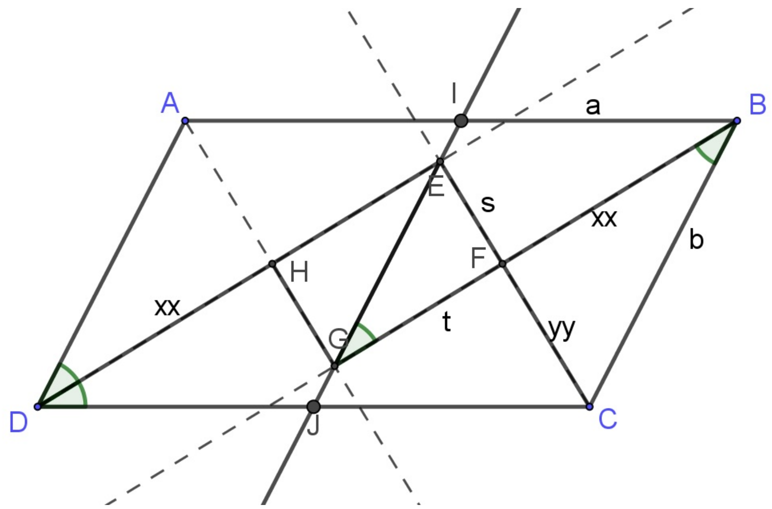
Solutions

1. **Solution 1.**



First note that $\triangle BFC \sim \triangle DEC$ (by AA), so $\dfrac{yy}{s+yy} = \dfrac{xx}{t+xx}$. Here $BF = DH = xx$ and $FC = yy$. (This can also be obtained by expressing $\tan(\hat{B}/2)$ in the two triangles.)

Cross-multiplying we get $\dfrac{s}{t} = \dfrac{yy}{xx}$, so $\angle FGE \cong \angle FBC$.

We also need the length of the segment EG. Notice that $\triangle BGI$ is isosceles, so $GI = IB = a/2$. Also $\triangle DEJ$ is isosceles, so $EJ = DJ = a/2$. It follows that

$$a = a/2 + a/2 = GI + EJ = b + EG, \text{ so } EG = |a-b|.$$

These allow us to compute:

$$\begin{aligned}\frac{\text{Area}(ABCD)}{\text{Area}(EFGH)} &= \frac{ab\cdot\sin(\hat{B})}{st} = \frac{ab\cdot 2\sin(\hat{B}/2)\cos(\hat{B}/2)}{\tan(\hat{B}/2)t^2}\\ &= \frac{2ab}{(t/\cos(\hat{B}/2))^2} = \frac{2ab}{EG^2} = \frac{2ab}{(a-b)^2}.\end{aligned}$$

Solution 2. Let the $AB = CD = a$, $BC = DA = b$, and $\alpha = \frac{1}{2}\angle DAB = \frac{1}{2}\angle BCD$, then

$$EF = |CE - CF| = |a\cos\alpha - b\cos\alpha| = |a-b|\cos\alpha$$

and

$$FG = |BF - BG| = |b\sin\alpha - a\sin\alpha| = |a-b|\sin\alpha.$$

So

$$\text{Area}(EFGH) = (a-b)^2 \cos\alpha \sin\alpha = \frac{1}{2}(a-b)^2 \sin 2\alpha.$$

On the other hand, $\text{Area}(ABCD) = ab \sin 2\alpha$ and then the result follows.

2. Let the length, width, and the height of the rectangular box be a, b, c inches, respectively. Then

$$4(a+b+c) = 4\ell, \quad 2(ab+bc+ac) = 32, \quad abc = 8,$$

which simplifies to

$$a+b+c = \ell, \quad ab+bc+ac = 16, \quad abc = 8.$$

Therefore a, b, c are the three roots of the equation $f(x) = 0$ with

$$f(x) = x^3 - \ell x^2 + 16x - 8.$$

(i) When $\ell = 9$, we observe that $f(1) = 0$, then we can factor $f(x)$ as $(x-1)(x^2 - 8x + 8)$ and find the three roots 1, $4 - 2\sqrt{2}$, and $4 + 2\sqrt{2}$. All these three roots are positive and they are the values for the length, the width, and the height of the box in inches.

(ii) If $\ell = 10$, then we can write

$$f(x) = x^3 - 10x^2 + 16x - 8 = x(2-x)(8-x) - 8.$$

If $f(x)$ has three positive roots a, b, c, then one of the roots must be no more than 2 because $abc = 8$. Without loss of generality, suppose that $0 < a \leq 2$. Let $g(a) = a(2-a)$, then $g(a) \geq 0$ and the maximum value 1 of $g(a)$ is reached when $a = 1$; hence $0 \leq g(a) \leq 1$ when $a \in (0, 2]$. Observing that $8 - a < 8$, it follows that

$$f(a) = g(a)(8-a) - 8 < (1)(8) - 8 = 0,$$

which is a contradiction. Therefore $f(x)$ can not have three positive roots, and we conclude that it is impossible to construct such a box.

Remark. *We can verify that $\ell = 8$ also works; in fact, 8 and 9 are the only integer values for ℓ to make the construction of such a box possible.*

3. **Solution 1.**

(i) This follows from simple algebra:

$$x_1^2 + x_2^2 + x_3^2 = a^4 + a^2b^2 + b^2c^2 = a^2(a^2+b^2) + b^2c^2 = (a^2+b^2)c^2 = (c^2)^2.$$

(ii) Consider a Pythagorean triple (a, b, c). Extend the pattern from part (i) to see that the following is a Pythagorean 5-tuple:

$$(a^3, a^2b, abc, bc^2, c^3).$$

(iii) Extend the pattern from part (i) and (ii) to see that the following is a Pythagorean 6-tuple:

$$(a^4, a^3b, a^2bc, abc^2, bc^3, c^4)$$

Solution 2.

(ii) Let (a, b, c) be a Pythagorean 3-tuple, then we have $a^2 + b^2 = c^2$, which if we square both sides, gives $(a^2)^2 + (ab)^2 + (ab)^2 + (b^2)^2 = c^4$, so (a^2, ab, ab, b^2, c^2) is a 5-P-tuple.

(iii) Starting from a 4-P-tuple (a, b, c, d) with $a^2 + b^2 + c^2 = d^2$, then

$$\begin{aligned} d^4 &= d^2(a^2+b^2+c^2) = a^2d^2 + b^2d^2 + c^2d^2 = a^2(a^2+b^2+c^2) + b^2d^2 + c^2d^2 \\ &= (a^2)^2 + (ab)^2 + (ac)^2 + (bd)^2 + (cd)^2, \end{aligned}$$

so $(a^2, ab, ac, bd, cd, d^2)$ is a 6-P-tuple.

Remark. Solution 2 can be generalized to get:

(1') if we have an m-P-tuple, and an n-P-tuple, then we can generate a $[(m-1)(n-1)+1]$-P-tuple (we use $m = n = 3$ for the 5-P-tuple above);

(2') if we have an n-P-tuple, we can generate a $(2n-2)$-P-tuple $(m, n \geq 3)$ (we used n=4 for the 6-P-tuple above).

By the mathematical induction, we can show that this way we can generate an n-P-tuple for any $n \geq 3$.

For example, to generate a 7-P-tuple, we choose $m = 3$, $n = 4$; starting from a 3-P-tuple (x, y, z), and a 4-P-tuple (a, b, c, d), we have a 7-P-tuple as $(xa, xb, xc, ya, yb, yc, zd)$.

Solution 3. From a Pythagorean n-tuple $(x_1, x_2, x_3, \ldots, x_{n-1}, x_n)$ one can construct the Pythagorean $(n+1)$-tuple $(3x_1, 4x_1, 5x_2, 5x_3, \ldots, 5x_{n-1}, 5x_n)$. We could do this using any Pythagorean 3-tuples to move from n-tuples to $(n+1)$-tuples.

4. (i) For the seed $(2, 2019)$ we generate the sequence

$$2, 2019, 2020, \frac{4038}{2018}, \frac{2020}{2018}, 2, 2019, 2020, \ldots.$$

It follows that the sequence exhibits a periodicity of length 5, and, in particular, $x_7 = 2019$.

(ii) If we start with the seed $x_1 = a$ and $x_2 = b$, then a computation shows that

$$\begin{aligned} x_3 &= \frac{b+a-1}{a-1} \\ x_4 &= \frac{ab}{(a-1)(b-1)} \\ x_5 &= \frac{b+a-1}{b-1} \\ x_6 &= a \\ x_7 &= b \end{aligned}$$

From this it follows that $x_{n+5} = x_n$ for all $n \geq 1$. That is, the sequence x_n is periodic with period 5. Now if we know that $x_{2020} = 2020$ if and only if $x_5 = 2020$. This is equivalent to

$$\frac{b+a-1}{b-1} = 2020 \quad \Rightarrow \quad 2019(b-1) = a.$$

Now, for any positive integer n we can choose the integer seed $(2019n, n+1)$ and we will have $x_{2020} = 2020$. This implies the result.

(iii) For an integer seed (a, b), we have that $x_{2019} = 2019$ if and only if

$$2019 = x_{2019} = x_{2014} = \cdots = x_4 = \frac{ab}{(a-1)(b-1)}$$

Since $n-1$ and n are relatively prime, the only way for the right hand side of the previous equation to be an integer is if $k(a-1) = b$ and $l(b-1) = a$ for some integers k and l. This would force $kl = 2019 = 3 \cdot 673$. Checking that none of $k, l = 1, 3, 673$ or 2019 yield solutions, we conclude that there are no integer seeds for which $x_{2019} = 2019$.

Solution 2.

(iii) Following part (ii), $x_{2019} = 2019$ if and only if $x_4 = \frac{ab}{(a-1)(b-1)} = 2019$. On the other hand, if a, b are integers, then

$$\left|\frac{a}{a-1}\right| = \left|1 + \frac{1}{a-1}\right| \leq 1 + \frac{1}{|a-1|} \leq 2$$

and similarly,

$$\left|\frac{b}{b-1}\right| \leq 2.$$

Hence $|x_4| \leq 4$, so x_4 cannot be 2019 and neither can x_{2019}.

5. Before we present our solution let us see what is the main idea. Suppose that person 2 is chosen in the first round. Then person 2 is eliminated and the remaining persons 1,3,4,5,6,7,8 see the amount in their pockets go up A_2 dollars. The pot now contains A_2 dollars. Now suppose that second person chosen is person 3. The amount of money in his pocket is $A_3 + A_2$. So his money goes into the pot and the pot now contains $A_3 + 2A_2$ dollars. The remaining players pocket money goes up by $A_3 + A_2$ dollars.

   Let a_i, for $i = 1$ to 8 be amount of money that the person selected has in his pocket at the beginning of the ith round. Let A'_i be the **original** amount of money in the selected person's pocket. Let M_i be the amount of money in the pot at the end of the ith round. So, $a_0 = M_0 = 0$, $a_1 = A'_1$, and $a_i = A'_i + M_{i-1}$.

   So,

   (i) $M_1 = a_1 = A'_1$.

   (ii) $M_2 = a_2 + M_1 = A'_2 + M_1 + M_1 = A'_2 + 2M_1 = A'_2 + 2A'_1$.

   (iii) $M_3 = a_3 + M_2 = A'_3 + 2M_2 = A'_3 + 2A'_2 + 4A'_1$.

   (iv) $M_4 = A'_4 + 2A'_3 + 2^2A'_2 + 2^3A'_1$.

   (v) $M_5 = A'_5 + 2A'_4 + 2^2A'_3 + 2^3A'_2 + 2^4A'_1$.

   (vi) $M_6 = A'_6 + 2A'_5 + 2^2A'_4 + 2^3A'_3 + 2^4A'_2 + 2^5A'_1$.

   (vii) $M_7 = A'_7 + 2A'_6 + 2^2A'_5 + 2^3A'_4 + 2^4A'_3 + 2^5A'_2 + 2^6A'_1$.

   (viii) So at the end of the seventh round, the value of the pot is

   $$M = A'_7 + 2A'_6 + 2^2A'_5 + 2^3A'_4 + 2^4A'_3 + 2^5A'_2 + 2^6A'_1.$$

   The remaining player takes home:

   $$a_8 = A'_8 + 2M_7 = A'_8 + 2(A'_7 + 2A'_6 + 2^2A'_5 + 2^3A'_4 + 2^4A'_3 + 2^5A'_2 + 2^6A'_1)$$

   We calculate $\mathbb{E}(M)$. The value of $\mathbb{E}(a_8)$ is computed similarly. Note that

   $$\mathbb{E}(A_i) = \frac{A_1 + A_2 + A_3 + A_4 + ... + A_8}{8} = W.$$

   So, $\mathbb{E}(M) = W + 2W + 2^2W + ... + 2^6W$.

   Thus, $\mathbb{E}(M) = (2^7 - 1)W$.

   And, $\mathbb{E}(a_8) = (2^8 - 1)W$.