MMPC STUDENT CODE

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THE SIXTY-THIRD ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America

Part II

Wednesday, December 4, 2019

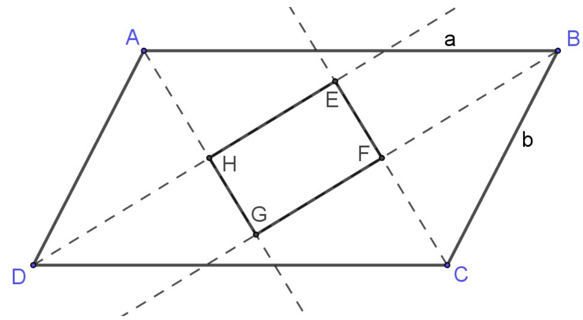
INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. Indicate whether we may use your e-mail address to contact you. **BUT DO NOT ACTUALLY WRITE YOUR NAME OR E-MAIL ADDRESS ON THIS BOOKLET.**
2. Part II consists of problems and proofs. You will be allowed 100 minutes (1 hour and 40 minutes) for the five questions. To receive full credit for a problem, you are expected to justify your answer.
3. You are not expected to solve all problems completely. Look over all the problems and work first on those that interest you the most. If you are unable to solve a particular problem, partial credit might be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.
4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the blank page at the end of the booklet (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page clearly write the **problem number**. If you use additional paper for your answer, check the appropriate box and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.
5. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may actually be easier to prove than the problem as stated.
6. The competition rules prohibit you from asking questions of anyone during the examination. The use of notes, reference material, computation aids, or any other aid is likewise prohibited. Please note that **calculators are not allowed** on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. Please do not return scratch paper containing routine numerical calculations.
7. You may now open the test booklet.

#	1	2	3	4	5	Total
Score						

1. Consider a parallelogram $ABCD$ with sides of length a and b, where $a \neq b$. The four points of intersection of the bisectors of the interior angles of the parallelogram form a rectangle $EFGH$. A possible configuration is given below.



Show that

$$\frac{\text{Area}(ABCD)}{\text{Area}(EFGH)} = \frac{2ab}{(a-b)^2}.$$

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2. A metal wire of length 4ℓ inches (where ℓ is a positive integer) is used as edges to make a cardboard rectangular box with surface area 32 square inches and volume 8 cubic inches. Suppose that the whole wire is used.

   (i) Find the dimension of the box if $\ell = 9$, i.e., find the length, the width, and the height of the box without distinguishing the different orders of the numbers. Justify your answer.

   (ii) Show that it is impossible to construct such a box if $\ell = 10$.

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3. A *Pythagorean n-tuple* is an ordered collection of counting numbers $(x_1, x_2, \ldots, x_{n-1}, x_n)$ satisfying the equation

$$x_1^2 + x_2^2 + \cdots + x_{n-1}^2 = x_n^2.$$

For example, $(3, 4, 5)$ is an ordinary Pythagorean 3-tuple (triple) and $(1, 2, 2, 3)$ is a Pythagorean 4-tuple.

(a) Given a Pythagorean triple (a, b, c) show that the 4-tuple (a^2, ab, bc, c^2) is Pythagorean.

(b) Extending part (a) or using any other method, come up with a procedure that generates Pythagorean 5-tuples from Pythagorean 3- and/or 4-tuples. Few numerical examples will not suffice. You have to find a method that will generate infinitely many such 5-tuples.

(c) Find a procedure to generate Pythagorean 6-tuples from Pythagorean 3- and/or 4- and/or 5-tuples.

Note. You can assume without proof that there are infinitely many Pythagorean triples.

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4. Consider the recursive sequence defined by $x_1 = a, x_2 = b$ and

$$x_{n+2} = \frac{x_{n+1} + x_n - 1}{x_n - 1}, \quad n \geq 1.$$

We call the pair (a, b) the *seed* for this sequence. If both a and b are integers, we will call it an *integer seed.*

(a) Start with the integer seed $(2, 2019)$ and find x_7.

(b) Show that there are infinitely many integer seeds for which $x_{2020} = 2020$.

(c) Show that there are no integer seeds for which $x_{2019} = 2019$.

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5. Suppose there are eight people at a party. Each person has a certain amount of money. The eight people decide to play a game. Let A_i, for $i = 1$ to 8, be the amount of money person i has in his/her pocket at the beginning of the game. A computer picks a person at random. The chosen person is eliminated from the game and their money is put into a pot. Also magically the amount of money in the pockets of the remaining players goes up by the dollar amount in the chosen person's pocket. We continue this process and at the end of the seventh stage emerges a single person and a pot containing M dollars. What is the expected value of M? The remaining player gets the pot and the money in his/her pocket. What is the expected value of what he/she takes home?

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(Continued Solutions)

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America

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