THE SIXTY-THIRD ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

Sponsored by

The Michigan Section of the Mathematical Association of America

Part I
(SOLUTIONS)

Tuesday, October 8, 2019

INSTRUCTIONS

1. Your answer sheet will be graded by machine. Carefully read and follow the instructions printed on the answer sheet. Check to ensure that your five-digit MMPC code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor asks you to stop, please quit working immediately and turn in your answer sheet.

3. Consider the problems and responses carefully. You may work out ideas on scratch paper before selecting a response.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.

5. For each of the questions, five different possible responses are provided. Choose the correct answer and completely fill in the corresponding bubble on your answer sheet.

6. Any scientific or graphing calculator is permitted on Part I. Unacceptable machines include computers, PDAs, pocket organizers, cell phones, and similar devices. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator. No other devices are permitted.

7. No one is permitted to explain to you the meaning of any question. Do not ask anyone to violate the rules of the competition. If you have questions concerning the instructions, ask them now.

8. You may open the test booklet and begin.

1. Suppose that A, B, and C are sets such that $|A| = 16$, $|A \cap C| = 4$, $|A \cap B| = 7$, and $|A \cap B \cap C| = 3$. Find the number of elements that are in A only.

(A) 2 (B) 8* (C) 9 (D) 11 (E) 12

Solution. Using a Venn digram one can see that the number of elements only in A is 8.

2. How many of the fractions

$$\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{19}, \frac{1}{20}$$

have a non-trivial repeating decimal expansion? (A repeating expansion where the repeating part consists of 0's or 9's is considered trivial, like in $\frac{1}{2} = 0.5000\cdots = 0.4999\ldots$)

(A) 10 (B) 11 (C) 12* (D) 13 (E) 14

Solution. If the denominator has a prime factorization which contains only twos and/or fives, then the decimal expansion will terminate and not repeat. Therefore, the seven numbers $1/2, 1/4, 1/5, 1/8, 1/10, 1/16$, and $1/20$ will all have terminating decimal expansions. Since there are 19 fractions in our list, there are 12 which have repeating decimal expansions.

3. How many ordered pairs (a, b) of positive integers are there such that $a \leq 5 \leq b$ and $a, 5, b$ are the lengths of three sides of a triangle?

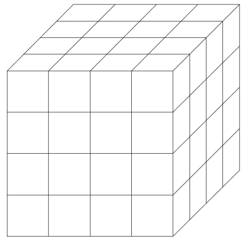
(A) 10 (B) 15* (C) 20 (D) 25 (E) 30

Solution. Each qualified pair (a, b) should satisfy the given inequality and also $a > 0$ and $a + 5 > b$. So they are

$$\begin{aligned}
&(1, 5)\\
&(2, 5), (2, 6)\\
&(3, 5), (3, 6), (3, 7)\\
&(4, 5), (4, 6), (4, 7), (4, 8)\\
&(5, 5), (5, 6), (5, 7), (5, 8), (5, 9)
\end{aligned}$$

The answer is (B).

4. A transparent plastic cube is painted red on all of its six sides, and then is divided into 64 equal-sized small cubes as illustrated below:



Randomly pick a small cube, such that each small cube is equally likely to be picked. What is the probability that the small cube has red paint exactly on two sides?

(A) $\frac{1}{8}$ (B) $\frac{3}{16}$ (C) $\frac{1}{4}$ (D) $\frac{3}{8}$* (E) $\frac{1}{2}$

Solution. All the small cubes with two sides red are along the edges and along each edge there are two of them. The total is $2 \times 12 = 24$, and then the desired probability is $24/64 = 3/8$. The answer is (D).

5. Consider a sequence $a_1, a_2, \ldots$ such that $a_{n+1} = a_1 + 2a_2 + 3a_3 + \ldots + na_n$, for all $n = 1, 2, \ldots$. (In other words, $a_2 = a_1$, $a_3 = a_1 + 2a_2$, $a_4 = a_1 + 2a_2 + 3a_3$, and so on.) If $a_{2019} = 2019$, find the value of a_{2018}.

(A) 1* (B) 2018 (C) 2019 (D) $\frac{2019}{2}$ (E) $\frac{2019}{2018}$

Solution. We have $2019 = a_{2019} = 2018a_{2018} + a_{2018}$, thus $a_{2018} = 1$. The answer is (A).

6. Denote by x_1, x_2, and x_3 the roots of $x^3 = 1$. Then, for all counting numbers n, the quantity $x_1^n + x_2^n + x_3^n$ equals ...

(A) 0 (B) 3 (C) $(x_1 + x_2 + x_3)^n$ (D) $x_1^n x_2^n + x_1^n x_3^n + x_2^n x_3^n$* (E) $x_1^n x_2^n x_3^n$

Solution. One of the roots, say x_1, equals 1, and from $x_1 x_2 x_3 = 1$ this implies that $x_2 x_3 = 1$. Consequently,

$$x_1^n x_2^n + x_1^n x_3^n + x_2^n x_3^n = x_2^n + x_3^n + 1 = x_1^n + x_2^n + x_3^n,$$

so the correct answer is (D).

7. Let ABC be a right triangle with hypotenuse c and legs a and b. Draw an altitude from the right angle to the hypotenuse. Which of the following formulae gives the correct expression for the length of this altitude?

(A) $\dfrac{a^{-1} + b^{-1}}{ab}$ (B) $\dfrac{\sqrt{ab}}{c}$ (C) $\dfrac{ab}{c}$* (D) $\dfrac{c}{ab}$ (E) $a^{-1} + b^{-1} + c^{-1}$

Solution. Let D be the point on the hypotenuse which is the foot of the altitude. Then ABC is similar to ACD by Angle-Angle similarity. It follows that

$$\frac{AB}{AC} = \frac{BC}{CD} \Rightarrow \frac{c}{b} = \frac{a}{CD}.$$

It follows that $CD = \dfrac{ab}{c}$.

8. Which of the following are rational numbers (numbers which can be expressed as the ratio of two integers)?

I. The only negative root of $p(x) = x^3 - 3x + 1$.

II. $\log_{35}(1 + 2 + \cdots + 49)$.

III. The only positive root of $q(x) = x^3 - 15x - 50$.

(A) II (B) III (C) I and II (D) I and III (E) II and III*

Solution. The Rational Root Theorem implies that the only possible rational roots of p are ± 1. Since $p(\pm 1) \neq 0$, there are no rational roots of p. We know that

$$1 + 2 + \cdots + 49 = \frac{50 \cdot 49}{2} = 5^2 \cdot 7^2,$$

so $\log_{35}(1 + 2 + \cdots + 49) = 2$. This implies that II is true. Finally, factoring q we see

$$q(x) = (x - 5)(x^2 + 5x + 10),$$

which implies that the only positive root of q is 5. This implies that III is true.

9. Let $x \vee y$ denote the larger of the values x and y and $x \wedge y$ denote the smaller of the values x and y. Suppose that we know $v < w < x < y < z$. What is

$$(((y \vee z) \wedge v) \vee ((v \vee x) \wedge w))) \wedge ((z \wedge y) \vee (w \vee y))?$$

(A) v (B) w* (C) x (D) y (E) z

Solution. First simplify the innermost parentheses,

$$(((y \vee z) \wedge v) \vee ((v \vee x) \wedge w))) \wedge ((z \wedge y) \vee (w \vee y)) = ((z \wedge v) \vee (x \wedge w)) \wedge (y \vee y)$$

Next simplify the new innermost parentheses,

$$((z \wedge v) \vee (x \wedge w)) \wedge (y \vee y) = (v \vee w) \wedge y = w \wedge y = w.$$

10. A robot has to move n ft, where n is a positive integer, in a straight direction. The robot is allowed two types of moves. A Type I move is where the robot moves 1 ft. A Type II move is where the robot moves 2 ft. Let R_n be the number of ways the robot can move n ft. Compute R_{10}.

(A) 45 (B) 68 (C) 88 (D) 89* (E) 90

Solution. Computing directly and using observation we see that $R_1 = 1$, $R_2 = 2$, $R_3 = 3$, $R_4 = 5$, and $R_5 = 8$. Following this Fibonacci pattern we get that $R_{10} = 89$.

11. Consider the sequence $a_1, a_2, \ldots,$ such that $a_1 = 1$ and $a_n = 2a_{n-1}$, for all integers $n > 1$. Compute the value of $\sum_{i=1}^{2019}(a_{i+1} - a_i)$.

(A) $2^{2019} - 1$* (B) 2^{2019} (C) $2^{2020} - 1$ (D) 2^{2020} (E) $2^{2021} - 1$

Solution. One can observe that

$$a_n = 2^{n-1}.$$

So our sum becomes

$$\sum_{i=1}^{2019}(a_{i+1} - a_i) = \sum_{i=1}^{2019} 2^{i-1} = 2^{2019} - 1.$$

12. Suppose that you have a deck of n cards numbered 1 through n, with n at least 3. Shuffle these cards so that the deck is in random order. What is the probability that the card labeled 1 is in the first, or $\left\lfloor \frac{n+1}{2} \right\rfloor$, or n position? (Here $\lfloor x \rfloor$ denotes the floor of x, that is, the greatest integer smaller or equal to x.)

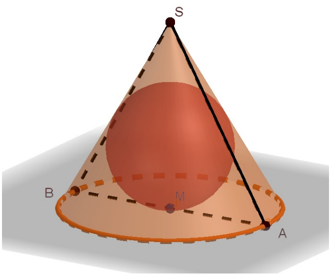
(A) $\frac{6}{n!}$ (B) $\frac{27}{n^3}$ (C) $\frac{1}{n}$ (D) $\frac{2}{n}$ (E) $\frac{3}{n}$*

Solution. Since there are n cards, there are $n!$ ways to shuffle them. The number of shuffles where card one is fixed in the first position is $(n-1)!$ Similarly, when card 1 is fixed in the $\left\lfloor \frac{n-1}{2} \right\rfloor$ and nth positions. Each of these probabilities is

$$\frac{(n-1)!}{n!} = \frac{1}{n}.$$

So the probability is $\frac{3}{n}$.

13. The picture below shows an equilateral triangle $\triangle SAB$ of side a which is an axial cross section of a right circular cone. Find the radius of the sphere inscribed in the cone.



(A) $\frac{a}{2\sqrt{3}}$* (B) $\frac{a}{2(1+\sqrt{3})}$ (C) $\frac{a}{6}$ (D) $\frac{a}{4}$ (E) $\frac{a\sqrt{3}}{4}$

Solution. The shown cross-section is through the center of the sphere, so it will intersect the sphere in a great-circle. The problem reduces to a plane geometry fact: find the radius of the inscribed circle in an equilateral triangle of side a. With trigonometry or by using Pythagorean theorem, the answer is (A).

14. A class of students took an exam consisting of two problems. 20 students solved problem 1, 19 students solved problem 2, and the number students that did not solve any of the two problems is 10 more than the number of students that solved both problems. Find the number of students in the class.

(A) less than 29 (B) 29 (C) 39 (D) 49* (E) more than 49

Solution. Let A be the set of students that solved problem 1, and B the set of students that solved problem 2, and $x = |A \cap B^c|$, $y = |B \cap A^c|$, $z = |A \cap B|$, $w = |(A \cup B)^c|$, where for any set X, X^c denotes the complement of X. Then

$$x + z = 20, \quad y + z = 19, \quad w = z + 10.$$

Adding the three equations we have $x + y + z + w = 49$, which is the number of students in the class. The answer is (D).

15. Assume that $a, x, y > 1$. Which of the following expressions equals $\log_{xy} a$?

(A) $\log_x a + \log_y a$ (B) $\log_x(ay) - \log_y(ax)$ (C) $\frac{\log_x a}{1 + \log_x y}$*

(D) $\frac{\log_y a}{1 - \log_x y}$ (E) $\frac{\log_y a}{\log_a x + \log_x a}$

Solution. Let $z = \log_{xy} a$. Then $(xy)^z = a$. Take the log base x of both sides of the equation:

$$\log_x((xy)^z) = \log_x(a) \Rightarrow= z(1 + \log_x y) = \log_a x.$$

It follows that $z = \dfrac{\log_x a}{1 + \log_x y}$.

16. Suppose that the function f satisfies the functional identity

$$f(x) + f\,(\min(x + 1, 7 - 2x)) = x$$

for every real number x. (The minimum of a and b, $\min(a, b)$, equals a when a is less than or equal to b and equals b when b is less than a.) Find $f(2)$.

(A) 0* (B) 1 (C) 2 (D) −1 (E) −2

Solution. Plugging in $x = 1, 2, 3$ we see that

$$f(1) + f(2) = 1 \quad (1)$$
$$f(2) + f(3) = 2 \quad (2)$$
$$f(3) + f(1) = 3 \quad (3)$$

The first and third equation imply that $f(3) - f(2) = 2$. The second equation now implies that $f(3) = 2$ and $f(2) = 0$.

17. Consider a regular pentagon with side lengths 1. Connect the midpoints of the sides of the pentagon to form another pentagon. What is the ratio of the area of the smaller pentagon to the area of the larger pentagon?

(A) $\cos^2(3\pi/10)$ (B) $\sin^2(3\pi/10)$* (C) $\frac{1}{2}$
(D) $\sin^2(3\pi/5)$ (E) $\cos^2(3\pi/5)$

Solution. The interior angles of a regular pentagon are $3\pi/5$. To find the area of a regular pentagon with side length L, break the pentagon into five isosceles triangles with angles $3\pi/10, 3\pi/10, 2\pi/5$. It follows that the area is

$$\frac{5}{4}L^2 \tan(3\pi/10).$$

Now connecting the midpoints of the sides of a regular pentagon will result in another regular pentagon with side lengths $\sin(3\pi/10)$. It follows that the ratio of the area of the smaller pentagon to the larger pentagon is

$$\frac{\frac{5}{4}\sin^2(3\pi/10)\tan(3\pi/10)}{\frac{5}{4}\tan(3\pi/10)} = \sin^2(3\pi/10).$$

18. Suppose that line L_1 and line L_2 are perpedicular. Let m_1 be the slope of L_1 and let m_2 be the slope of line L_2. Suppose that the square of the slope of line L_1 plus eight times the square of the slope of L_2 is 6. For the rational values of m_1 and m_2 compute the quantity $\left(\frac{1}{m_1}\right)^2 + 2m_2^2$.

(A) $\frac{1}{2}$ (B) $\frac{3}{4}$* (C) $\frac{3}{2}$ (D) $\frac{27}{16}$ (E) $\frac{17}{2}$

Solution. Since L_2 and L_1 are perpendicular then $m_2 = \frac{-1}{m_1}$. Then the condition

$$m_1^2 + 8m_2^2 = 6$$

becomes

$$m_1^4 - 6m_1^2 + 8 = 0.$$

The rational roots of this equation are $m_1 = 2$ and $m_1 = -2$. Using either of these gives that

$$\left(\frac{1}{m_1}\right)^2 + 2m_2^2 = \frac{3}{4}$$

19. Suppose that x, y, and z are positive real numbers for which $x+y+z = 1$ and $\frac{z}{x+y} = \frac{x}{y+z} = \frac{y}{x+z}$. Find xyz.

(A) 1/27* (B) $\sqrt{2}/27$ (C) 16/81 (D) $\sqrt{3}/27$ (E) 8/27

Solution. Since the three ratios are equal, notice that the sum of the three numerators and the sum of the three denominators have the same ratio. We obtain:

$$\frac{z}{x+y} = \frac{x}{y+z} = \frac{y}{x+z} = \frac{z+x+y}{x+y+y+z+x+z} = \frac{1}{2}.$$

Hence $2z = x+y$ and therefore $z = 1/3$ from the equation $x+y = 1-z$. The symmetry between x, y, and z implies that $x = y = 1/3$ also, so it follows that $xyz = 1/27$.

20. A number N is called a *triplet* if it can be written in base b as aaa_b. Find the sum of all triplets for bases b which satisfy $2 \le b \le 5$. Give your answer in base 10.

(A) 356 (B) 383 (C) 425 (D) 482* (E) 512

Solution. We calculate the sum of all triplets as:

$$\sum_{b=2}^{5}\sum_{a=1}^{b-1}(ab^2+ab+a)=\sum_{b=2}^{5}\frac{b(b-1)}{2}(b^2+b+1)=\frac{1}{2}\sum_{b=2}^{5}(b^4-b)=$$

$$\frac{1}{2}(2^4+3^4+4^4+5^4-2-3-4-5)=482.$$

21. A special type of door lock has a panel with three buttons labeled with the digits 1, 2 and 3. The lock is opened by a sequence of two actions. Each action consists of either pressing one of the buttons or pressing two of them simultaneously. For example (12)(3) is a possible combination. Another possible combination is (1)(2). Note that (12)(3) and (21)(3) are the same combination, since (12) and (21) refer to pressing at the same time the buttons 1 and 2. How many possible lock combinations are there?

(A) 9 (B) 18 (C) 27 (D) 36* (E) 72

Solution. Let action one be denoted by A_1, where A_1 is action of pressing two buttons simultaneusly. Let A_2 be the action where one button is pushed. So the possible combinations are A_1A_1, A_1A_2, A_2A_1 and A_2A_2. There 9 possible combinations for each. Hence, 36 possible combinations.

22. Consider the function $f(x)=\frac{1}{x}$. For $a>0$, denote by L the line with slope $-\frac{1}{a^2}$ that intersects $f(x)$ at the single point $(a, f(a))$. Compute the area of triangle formed by L and the coordinate axes.

(A) 4 (B) 2* (C) 1 (D) $\frac{4}{a}$ (E) $\frac{1}{2a^2}$

Solution. The line described has the equation

$$y=\frac{-1}{a^2}x+\frac{2}{a}.$$

So our triangle bounded by the curve $y=\frac{1}{x}$ has height $\frac{2}{a}$ and base $2a$. So the area is

$$\frac{1}{2}(2a)(\frac{2}{a})=2.$$

23. In a urn there are 5 red balls, 3 white balls, and 2 black balls. One randomly chooses n balls from the urn. Find n such that the probability that among the chosen n balls there is at least one black ball is strictly bigger than $\frac{8}{15}$.

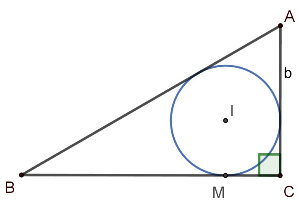
(A) $\{9,10\}$ (B) $\{8,9,10\}$ (C) $\{6,7,8,9,10\}$ (D) $\{4,5,6,7,8,9,10\}$* (E) any n larger than 2 will do

Solution. Using the interpretation of probability as number of favorable events over the total number of occurrences, the probability to get at least one black ball is

$$\frac{C_{8,n-2}+2C_{8,n-1}}{C_{10,n}}=\frac{1+\frac{2(10-n)}{n-1}}{\frac{90}{n(n-1)}}=\frac{n(19-n)}{90}.$$

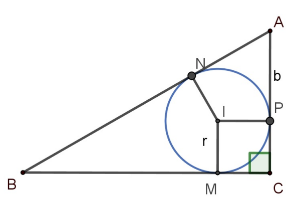
Asking this to be **strictly** larger than $8/15$, leads to $n\geq 4$, so the answer is (D).

24. Consider a right triangle $\triangle ABC$, with $\angle C=90^\circ$ and $\angle A=\frac{\angle B+\angle C}{2}$. Denote $AC=b$ and let M be the tangency point of $\overline{BC}$ with the inscribed circle. Find $BM\cdot MC$.



(A) $\frac{b}{\sqrt{3}}$ (B) $\frac{b}{1+\sqrt{3}}$ (C) $\sqrt{3}b^2$ (D) b^2 (E) $\frac{b^2}{2}$*

Solution. First it is clear that $\angle A=60^\circ$. Denote the radius of the inscribed circle by r. From the formula $r\cdot(\text{semi-perimeter})=\text{area triangle}$, we get $r=b/(1+\sqrt{3})=b(\sqrt{3}-1)/2$. In the picture below, we have $IM=IN=IP=r$.



Consequently:

$$\begin{aligned} BM \cdot MC = BM \cdot IM &= \text{Area}(BMIN) \\ &= \text{Area}(\triangle ABC) - \text{Area}(CPIM) - \text{Area}(ANIP) \\ &= \text{Area}(\triangle ABC) - 2\text{Area}(\triangle AIC) \\ &= b^2\sqrt{3}/2 - AC \cdot IP = b^2\sqrt{3}/2 - b^2(\sqrt{3}-1)/2 = b^2/2. \end{aligned}$$

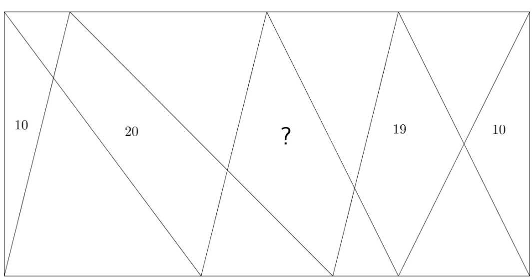
So the correct answer is (E).

25. The number 14641 has the property that it is a perfect square when interpreted in any base $b \geq 7$. Assuming $b = 7$, find the number x such that $x^2 = 14641$. Here 14641 is written in base 7, but your answer should be given in base 10.

(A) 8 (B) 50 (C) 64* (D) 101 (E) 121

Solution. $14641 = 121^2$ and $121_7 = 64_{10}$. The answer is (C).

26. A rectangle is divided into various parts by segments with end points on its sides, with areas of four parts marked, as in the graph below.



Find the area of the part with the question mark "?".

(A) $\sqrt{19 \times 20}$ (B) 19* (C) 19.5 (D) 20 (E) 21

Solution. Starting from the lower-left corner in the counterclockwise direction we label the points in the graph along the boundary of the rectangle as $A, P, Q, R, B, C, Z, Y, X, D$. Then the sum of the areas of triangles AXQ and QZB is the same as the sum of the areas of triangles ADP, PYR and RCB. It follows that $10 + ? + 10 = 20 + 19$, which gives $? = 19$. The answer is (B).

27. How many ordered pairs (a, b) of real numbers are there such that $(2a^2+1)+(2a^2-1)\mathbf{i}$ is a solution to the equation $x^2-10x+b^2+4b+20=0$ (where $\mathbf{i}$ is the imaginary unit)?

(A) 1 (B) 2 (C) 4 (D) 6* (E) 8

Solution. There are two cases: (i) $2a^2-1=0$. We have two values for a in this case, and $2a^2+1=2$ is a solution to the equation, which gives $b^2+4b+4=0$ and one value for b, so we obtain two pairs of (a, b). (ii) $2a^2-1\neq 0$. In this case $(2a^2+1)-(2a^2-1)\mathbf{i}$ is also a solution to the equation. So the sum of the two roots must be $-(-10)/1=10$ due to Vieta's Theorem, which gives $2a^2+1=5$ and two values for a. We further get $2a^2-1=3$ and the two roots are $5\pm 3\mathbf{i}$. Since the product of the two roots is $b^2+4b+20$, this gives $b^2+4b-14=0$ and two values for b. We obtain 4 pairs of (a, b) in this case. So the answer is (D). Note that we don't really need to get the actual values of a and b.

28. Let $\log x$ represent the common logarithmic function. How many ordered 3-tuples (a, b, c) of positive integers are there such that $\log(a+b+c)=\log a+\log b+\log c$?

(A) 0 (B) 1 (C) 6* (D) 24 (E) infinitely many

Solution. We first find (a, b, c) with $a\le b\le c$. In this case we have $abc=a+b+c\le 3c$ and then $ab\le 3$. The only possible (a, b) then are $(1, 1), (1, 2)$, and $(1, 3)$. We can check that only the pair $(1, 2)$ gives a satisfactory value for c, which is 3. So there is only one solution $(1, 2, 3)$ with $a\le b\le c$. There are 6 permutations of the 3-tuple, which yields the answer (C).

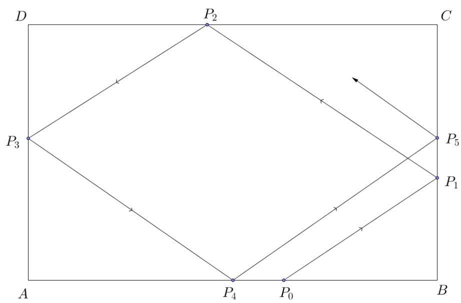
29. What is the probability that all of the spades are next to each other in an ordinary deck of playing cards?

(A) $\frac{13!}{40!}$ (B) $\frac{13!}{52!}$ (C) $\frac{4\cdot 13!}{52!}$ (D) $\frac{4\cdot 40!}{52!}$ (E) $\frac{13!\cdot 40!}{52!}$*

Solution. Group all the spades together to form one group. There are 13! ways of arranging this group. There are 40! ways of arranging the grouping of the spades and the other cards. Since there are 52! ways of arranging the deck, the answer is:

$$\frac{13!40!}{52!}.$$

30. A ball is moving inside a rectangle $ABCD$ and is bounced/reflected by the sides. It starts from the point P_0 on AB, and reaches side BC at P_1, then reaches side CD at P_2, etc, as illustrated in the picture below. (Suppose that the ball always reaches the interior of the adjacent side for its next bounce.)



The coordinates of A, B, C, D are $(0,0)$, $(18,0)$, $(18,12)$, $(0,12)$, respectively. If $P_0 = (3,0)$ and $P_4 = P_0$, find P_{2019}.

(A) $(0,2)$* (B) $(0,10)$ (C) $(3,12)$ (D) $(15,12)$
(E) different from (A)–(D)

Solution. Let $AB = a$, $BC = b$, $P_0B = x$ and $\angle P_1P_0B = \alpha$. Then $\angle P_0P_1B = \angle CP_1P_2 = \angle P_2P_3D = \angle AP_3P_4 = 90^\circ - \alpha$ and $\angle P_1P_2C = \angle P_3P_2D = \angle P_3P_4A = \alpha$. Since $P_4 = P_0$, the sequence $P_0, P_1, P_2, \ldots$ is periodic with period 4; so $P_{2019} = P_3$. Moreover,

$$\begin{aligned} BP_1 &= x\tan\alpha, \\ CP_2 &= (b - BP_1)\tan(90^\circ - \alpha) = (b - x\tan\alpha)/\tan\alpha, \\ DP_3 &= (a - CP_2)\tan\alpha = a\tan\alpha - b + x\tan\alpha, \\ AP_4 &= (b - DP_3)\tan(90^\circ - \alpha) = (b - DP_3)/\tan\alpha \\ &= (2b - a\tan\alpha - x\tan\alpha)/\tan\alpha. \end{aligned}$$

By $P_4 = P_0$ again, we obtain

$$a - x = AP_4 = (2b - a\tan\alpha - x\tan\alpha)/\tan\alpha,$$

which gives $\tan\alpha = b/a$. It follows that the orbit of the ball is a parallelogram with sides parallel to the diagonals of $ABCD$. Because $P_3A/AP_0 = b/a = 12/18$ and $AP_0 = 3$, we get $P_3A = 2$. So the answer is (A).

31. Find the area of the region defined by the set of points

$$S = \{(x,y) : 2019 \le \max(2|x|, 3|y|) \le 2020\}.$$

(The maximum of a and b, $\max(a,b)$, equals a when a is greater than or equal to b and equals b when b is greater than a.)

(A) $4(3\cdot 2020 - 2\cdot 2019)$ (B) $\frac{2}{3}(2020^2 - 2019^2)$* (C) $4(\frac{2020^2}{3} - \frac{2019^2}{2})$
(D) $2\cdot 2020^2 - 3\cdot 2019^2$ (E) $\frac{2020\cdot 2019}{6}$

Solution. By symmetry, we only need to consider the area in the first quadrant where $x, y \geq 0$. The portion of this region in the first quadrant can be written as the union of two rectangles:

$$\{(x,y) : 0 \leq x \leq \frac{2019}{2} \ \& \ \frac{2019}{3} \leq y \leq \frac{2020}{3}\} \cup$$

$$\{(x,y) : \frac{2019}{2} \leq x \leq \frac{2020}{2} \ \& \ 0 \leq y \leq \frac{2020}{3}\}.$$

The areas of these rectangles is

$$\frac{2019}{2} \cdot \left(\frac{2020}{2} - \frac{2019}{3}\right) + \frac{2020}{3} \cdot \left(\frac{2020}{2} - \frac{2019}{3}\right) = \frac{1}{6}(2020-2019)(2020+2019).$$

Quadrupling this we see that the area of the region is

$$\frac{2}{3}(2020^2 - 2019^2).$$

32. There are four large groups of people, each with 1000 members. Any two of these groups have 100 members in common. Any three have 10 members in common. And there is one person in all four groups. All together, how many people are in these groups?

(A) 3221 (B) 3439* (C) 3617 (D) 3659 (E) 3827

Solution. Denote the four groups by A_1, A_2, A_3 and A_4. Then we are trying to find the size of $|A_1 \cup A_2 \cup A_3 \cup A_4|$. Using the inclusion-exclusion principle:

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = 4(1000) - 6(100) + 4(10) - 1.$$

Hence,

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = 3439.$$

33. We say that x_0 is a *local maximum* of a function $f : \mathbb{R} \to \mathbb{R}$ if there exists an interval $I = (x_0 - a, x_0 + a)$ centered at x_0 such that $f(x_0) \geq f(x)$ for all $x \in I$. Find a condition on the coefficients b and c such that the function $f(x) = |-x^2 + bx + c|$ does not admit a local maximum.

(A) $b = 0$, no condition on c (B) $c = 0$, no condition on b
(C) $b^2 + 4c > 0$ (D) $b^2 + 4c \leq 0$*
(E) $f(x)$ has a local maximum for any choice of b and c

Solution. The condition is that the quadratic function $g(x) = -x^2 + bx + c$ (which opens downwards), has its vertex on the x-axis. We get $b^2 - 4a(-1) \leq 0$ or equivalently $b^2 + 4c \leq 0$, so the correct answer is (D).

34. Two balls are drawn from an urn containing n balls numbered 1 through n. Assume $n > 1$. The first ball is kept if it is numbered 1 and returned otherwise. What is the probability of the second ball being numbered 2?

(A) $\frac{2n-1}{2n(n-1)}$ (B) $\frac{1}{n}$ (C) $\frac{1}{n-1}$ (D) $\frac{n^2-n+1}{n^2(n-1)}$* (E) different from (A) to (D)

Solution. Using Bayes' theorem we see

$$P(2) = P(1)P(2|1) + P(\neg 1)P(2|\neg 1).$$

Note that $P(1) = \frac{1}{n}$, $P(\neg 1) = \frac{n-1}{n}$. So, under the rules of drawing the balls out of the urn,

$$P(2|1) = \frac{1}{n-1}$$

and

$$P(2|\neg 1) = \frac{1}{n}.$$

Putting this together gives

$$P(2) = \frac{n^2 - n + 1}{n^2(n-1)}.$$

35. If $\cos\theta + \sin\theta = \frac{1}{3}$ for some angle θ, what is the value of $\cos^3\theta + \sin^3\theta$?

(A) $\frac{1}{27}$ (B) $\frac{5}{27}$ (C) $\frac{13}{27}$* (D) $\frac{17}{27}$ (E) different from (A) to (D)

Solution. Square both sides of $\cos\theta + \sin\theta = \frac{1}{3}$, we obtain

$$\cos\theta\sin\theta = -\frac{4}{9}.$$

Then,

$$\cos^3\theta + \sin^3\theta = (\cos\theta + \sin\theta)^3 - 3\cos\theta\sin\theta(\cos\theta + \sin\theta) = \frac{13}{27}.$$

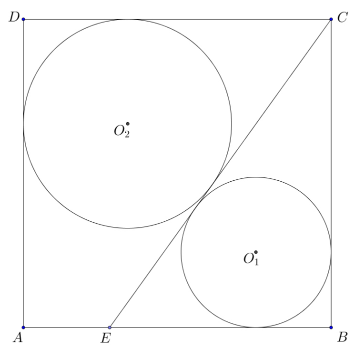
The answer is (C).

36. 36 houses are arranged in a square shape with 6 rows and 6 columns. They are to be painted either red or blue such that no three consecutive houses in a row are of the same color and no two consecutive houses in a column are of the same color. How many ways are there to paint these houses?

(A) 24^6 (B) 24 (C) 26^6 (D) 26* (E) 28

Solution. First we observe that a way of painting the houses is uniquely determined by the painting of the first row of the houses. Now we find the number of ways to paint the first row that there are three consecutive houses with the same color. Consider the first appearance of these three houses from the left: Case 1: The first three houses are of the same color; there are $2 \times 2^3 = 16$ choices. Case 2: The 2nd to the 4th houses are of the same color, the 1st is of a different color; there are $2 \times 2^2 = 8$ choices. Case 3: The 3rd to the 5th houses are of the same color, the 2nd house is of a different color; there are $2 \times 2^2 = 8$ choices. Case 4: The last three houses are of the same color, the 3rd is of a different color, and *the first three houses are not in the same color*; there are $2 \times 3 = 6$ choices. Overall there are 38 choices if there are three consecutive houses with the same color. Because $64 - 38 = 26$, the answer is (D).

37. Given a square $ABCD$ with side length 1 and a point E on the side AB, let O_1 be the circle that is tangent to the segments EB, BC, CE, and let O_2 be the circle that is tangent to the segments CD, DA, CE, as illustrated in the picture below.



If the radius of the circle O_1 is $\frac{1}{4}$, find the radius of the circle O_2.

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$* (C) $\frac{2}{5}$ (D) $\frac{1}{4}(7 - 4\sqrt{2})$ (E) $\frac{3}{4}(\sqrt{2} - 1)$

Solution. Let the lines CE and DA intersect at F, and $BE = x$. Suppose that circle O_1 is tangent to BC, BE, EC at P, Q, R, respectively. Then $BP = BQ = 1/4$ because PO_1QB is a square; then $PC = 1 - 1/4 = 3/4$, $QE = x - 1/4$. Then $EC = ER + CR = EQ + CP = x - 1/4 + 3/4 = x + 1/2$. Using $CE^2 = EB^2 + BC^2$, we obtain $(x + 1/2)^2 = x^2 + 1^2$, which derives $x = 3/4$. Because $\triangle CDF \sim EBC$, the ratio of the radius of circle O_2 to the radius of circle O_1 is $CD/EB = 1/(3/4) = 4/3$. Then the radius of circle O_2 is $(4/3)(1/4) = 1/3$. The answer is (B).

38. Solve the equation

$$3\sin^{-1}x + 2\cos^{-1}x + \tan^{-1}x = \frac{\pi}{2}.$$

(A) $\sqrt{\frac{-1+\sqrt{3}}{4}}$ (B) $-\sqrt{\frac{-1+\sqrt{3}}{4}}$ (C) $-\sqrt{\frac{1+\sqrt{5}}{2}}$
(D) $\sqrt{\frac{-1+\sqrt{5}}{2}}$ (E) $-\sqrt{\frac{-1+\sqrt{5}}{2}}$*

Solution. The co-function identity implies that $2\sin^{-1}x + 2\cos^{-1}x = \pi$, so the equation becomes

$$\sin^{-1}x + \tan^{-1}x = -\frac{\pi}{2}.$$

Take the cosine of this equation to get

$$\cos(\sin^{-1}x)\cos(\tan^{-1}x) - \sin(\sin^{-1}x)\sin(\tan^{-1}x) = 0.$$

$$\sqrt{1-x^2}\frac{1}{\sqrt{x^2+1}} - x\frac{x}{\sqrt{x^2+1}} = 0$$
$$\sqrt{1-x^2} - x^2 = 0.$$
$$x^4 + x^2 - 1 = 0.$$

It follows that

$$x^2 = \frac{-1\pm\sqrt{5}}{2} \quad \Rightarrow \quad x^2 = \frac{-1+\sqrt{5}}{2} \quad \Rightarrow \quad x = \pm\sqrt{\frac{-1+\sqrt{5}}{2}}.$$

The positive root is extraneous, since $\sin^{-1}x + \tan^{-1}x$ is negative. It follows that

$$x = x = -\sqrt{\frac{-1+\sqrt{5}}{2}}.$$

39. Consider the equation $2019 = 3^m - p\cdot 3^n$, with variables m, n, and p counting numbers. Find the smallest value of m that appears in a solution (m, n, p) of this equation.

(A) 5 (B) 6 (C) 7* (D) 8 (E) 10

Solution. Start with the factorization $2019 = 3\cdot 673$. After simplifying with 3, we are left to solving the equation $673 = 3^{m-1} - p\cdot 3^{n-1}$. As the right-hand-side contains a subtraction, the requested value is the smallest value m such that 3^{m-1} is bigger than 673, which is $m-1 = 6$. The answer is (C).

40. The equation $x^3 + px + q = 0$ has real roots x_1, x_2 and x_3 satisfying $\arctan(x_1) + \arctan(x_2) = \arctan(x_3)$. Assuming that $x_3 \neq 0$, find x_3.

(A) $2 - \frac{q^2}{4} - p$ (B) $2\sqrt{2-p}$ (C) $-\frac{q}{1+p}$ (D) $-\frac{p}{2}$ (E) $-\frac{q}{2}$*

Solution. Notice first that $x_1 + x_2 + x_3 = 0$. Apply the tangent function to the given equality to get:

$$\frac{x_1 + x_2}{1 - x_1 x_2} = x_3 = -(x_1 + x_2).$$

This can be further written:

$$(x_1 + x_2)\left[1 + \frac{1}{1 - x_1 x_2}\right] = 0.$$

Or $x_1 x_2 = 2$. Combined with $x_1 x_2 x_3 = -q$, we get the answer (E).

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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