

2015 Michigan Mathematics Prize Competition, Part 2: Solutions

1. Consider a right triangle with legs of lengths a and b and hypotenuse of length c such that the perimeter of the right triangle is numerically (ignoring units) equal to its area. Prove that there is only one possible value of $a + b - c$, and determine that value.

Solution: The Pythagorean Theorem gives $a^2 + b^2 = c^2$. So $a^2 + 2ab + b^2 - c^2 = 2ab$.

Thus $(a + b)^2 - c^2 = 2ab$. Factoring gives $(a + b + c)(a + b - c) = 2ab$.

Since the perimeter $a + b + c$ is numerically equal to the area $\frac{1}{2}ab$, we have $\frac{1}{2}ab(a + b - c) = 2ab$.

Since a and b are positive, we conclude that $a + b - c = 4$, and this is the only possible value.

Alternative solution: Since the perimeter is numerically equal to the area, we have $a + b + c = \frac{1}{2}ab$.

So $a + b = \frac{1}{2}ab - c$. Squaring both sides gives $a^2 + 2ab + b^2 = \frac{1}{4}a^2b^2 - abc + c^2$.

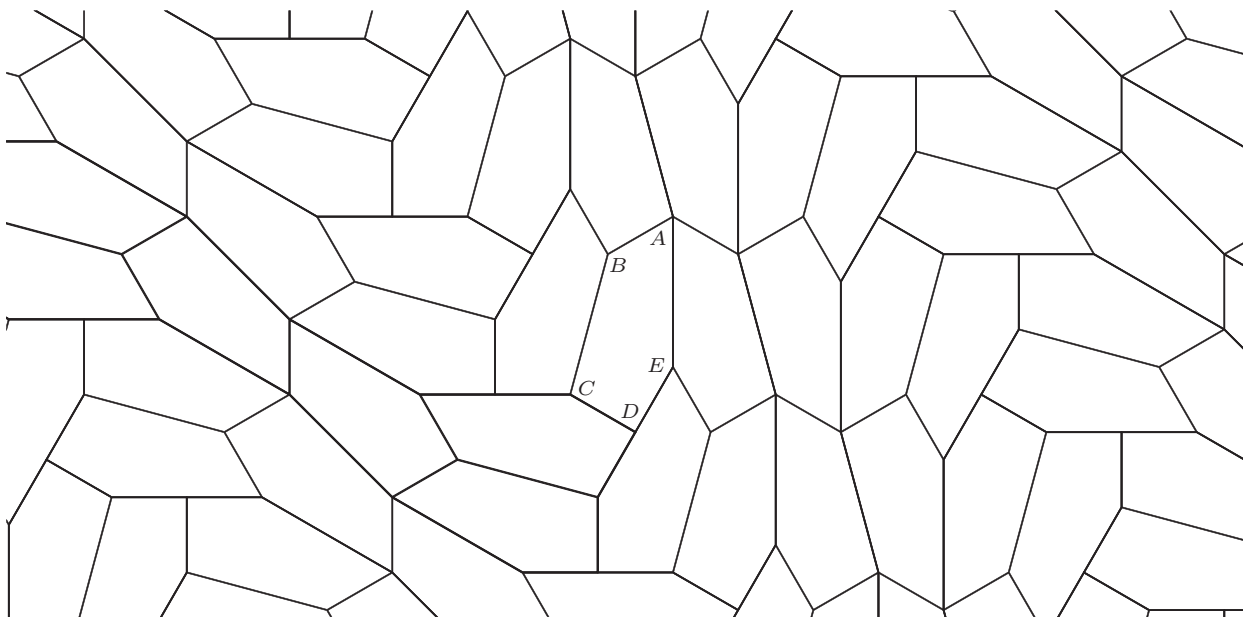
The Pythagorean Theorem allows us to cancel $a^2 + b^2$ on the left with c^2 on the right.

We are left with $2ab = \frac{1}{4}a^2b^2 - abc$.

Since a and b are positive, we can divide through by ab to get $2 = \frac{1}{4}ab - c$, or $4 = \frac{1}{2}ab - 2c$.

Substitute the perimeter $a + b + c$ for the area $\frac{1}{2}ab$ to conclude that $4 = a + b + c - 2c = a + b - c$, and this is the only possible value.

2. Last August, Jennifer McLoud-Mann, along with her husband Casey Mann and an undergraduate David Von Derau at the University of Washington, Bothell, discovered a new tiling pattern of the plane with a pentagon. This is the fifteenth pattern of using a pentagon to cover the plane with no gaps or overlaps. It is unknown whether other pentagons tile the plane, or even if the number of patterns is finite. Below is a portion of this new tiling pattern.



Determine the five angles (in degrees) of the pentagon $ABCDE$ used in this tiling. Explain your reasoning, and give the values you determine for the angles at the bottom.

Angle D is supplementary to itself, so $D = 90$.

Around B we have $2B + D = 360$, so $2B = 360 - D = 270$. So $B = 135$.

Around E we have $A + 2E = 360$. And around C we have $2C + E = 360$.

Also, the sum of the interior angles of a pentagon is $3 \cdot 180 = 540 = A + B + C + D + E$.

So $A + C + E = 540 - B - D = 540 - 135 - 90 = 315$.

Adding the three equations $A + C + E = 315$

$$\begin{array}{r} -A \qquad -2E = -360 \end{array}$$

$$2C + E = 360$$

gives $3C = 315$, so $C = 105$.

Thus $E = 360 - 2C = 360 - 210 = 150$. And finally, $A = 360 - 2E = 360 - 300 = 60$.

3. Let $f(x) = \sqrt{2019 + 4\sqrt{2015}} + \sqrt{2015}x$. Find all rational numbers x such that $f(x)$ is a rational number.

Solution: Note that $f(x) = \sqrt{2019 + 4\sqrt{2015}} + \sqrt{2015}x$
 $= \sqrt{2015 + 4\sqrt{2015} + 4} + \sqrt{2015}x$
 $= \sqrt{(\sqrt{2015} + 2)^2} + \sqrt{2015}x$
 $= \sqrt{2015} + 2 + \sqrt{2015}x$
 $= \sqrt{2015}(x + 1) + 2.$

Suppose this is rational $\frac{p}{q}$ where p and q are integers. Then $(x + 1)\sqrt{2015} = \frac{p-2q}{q}$. And unless $x = -1$, we have a contradiction that $\sqrt{2015} = \frac{p-2q}{q(x+1)}$ is rational if x is rational. Thus $x = -1$ is the only rational number for which $f(x)$ is rational.

4. Alice has a whiteboard and a blackboard. The whiteboard has two positive integers on it, and the blackboard is initially blank. Alice repeats the following process.

Let the numbers on the whiteboard be a and b , with $a \leq b$.
 Write a^2 on the blackboard.
 Erase b from the whiteboard and replace it with $b - a$.

For example, if the whiteboard began with 5 and 8, Alice first writes 25 on the blackboard and changes the whiteboard to 5 and 3. Her next move is to write 9 on the blackboard and change the whiteboard to 2 and 3.

Alice stops when one of the numbers on the whiteboard is 0. At this point the sum of the numbers on the blackboard is 2015.

- a. If one of the starting numbers is 1, what is the other?
 b. What are all possible starting pairs of numbers?

Solution, part a: Each line below indicates the current pair of number on the whiteboard and the new number written on the blackboard:

| step | whiteboard | blackboard |
|----------|-------------------|-------------|
| 0 | 1 b | — |
| 1 | 1 $b - 1$ | 1 |
| 2 | 1 $b - 2$ | 1 |
| \vdots | \vdots \vdots | \vdots |
| $b - 1$ | 1 1 | 1 |
| b | 0 1 | 1 |
| | | total= 2015 |

Thus the process has taken 2015 steps, and $b = 2015$.

Solution, part b: Notice that in going from a and b to a and $b - a$ on the whiteboard has reduced the product of these integers by a^2 , which is the number written on the blackboard. Thus each step preserves the quantity obtained by adding the numbers on the blackboard to the product of the two numbers on the whiteboard. After the final step, the sum of the numbers on the blackboard is 2015 and the product of the two numbers on the whiteboard is 0. So at the beginning when there are no numbers on the blackboard, the product of the two original numbers on the whiteboard must be 2015, Since $2015 = 5 \cdot 13 \cdot 31$, the possible starting numbers are 1, 2015; 5, 403; 13, 155; and 31, 65.

5. Professor Beatrix Quirky has many multi-volume sets of books on her shelves. When she places a numbered set of n books on her shelves, she doesn't necessarily place them in order with book 1 on the left and book n on the right. Any volume can be placed at the far left. The only rule is that, except the leftmost volume, each volume must have a volume somewhere to its left numbered either one more or one less. For example, with a series of six volumes, Professor Quirky could place them in the order 123456, or 324561, or 564321, but not 321564 (because neither 4 nor 6 is to the left of 5).

Let's call a sequence of numbers a *quirky sequence of length n* if:

1. the sequence contains each of the numbers from 1 to n , once each, and
2. if k is not the first term of the sequence, then either $k + 1$ or $k - 1$ occurs somewhere before k in the sequence.

Let q_n be the number of quirky sequences of length n . For example, $q_3 = 4$ since the quirky sequences of length 3 are 123, 213, 231, and 321.

- a. List all quirky sequences of length 4.
- b. Find an explicit formula for q_n . Prove that your formula is correct.

Solution, part a: 1234, 2134, 2314, 2341, 3214, 3241, 3421, 4321.

Solution, part b: Notice that a sequence is quirky if and only if after the first term, each term is either one greater than the largest number previously used or one less than the smallest number previously used. Thus a sequence is quirky if and only if, reading from right to left, each term is either the largest unused number or the smallest unused number. This gives two choices for each term except the first, which is forced to be the only remaining unused number. Thus $q_n = 2^{n-1}$.

Alternative solution, part b: Suppose inductively that $q_k = 2^{k-1}$ for $k = 1, 2, \dots, n-1$. We can form a quirky sequence of length n from a quirky sequence of length $n-1$ by inserting n into the sequence. Inserting n as the first term allows only 1 quirky sequence, namely $n, n-1, \dots, 2, 1$. Inserting n as the p th term requires the preceding terms to be a quirky sequence of length $p-1$ with $n-p$ added to each term (that is, $n-p+1, n-p+2, \dots, n-1$ in a quirky order). Thus, there are $1 + q_1 + q_2 + \dots + q_{n-1} = 1 + 1 + 2 + \dots + 2^{n-2} = 2^{n-1}$ quirky sequences of length n .