FIFTY-FIFTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part I

October 5, 2011

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Carefully read and follow the instructions printed on the answer sheet. Check to ensure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor asks you to stop, please quit working immediately and turn in your answer sheet.

3. Consider the problems and responses carefully. You may work out ideas on scratch paper before selecting a response.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.

5. For each of the questions, five different possible responses are provided. In some cases the fifth alternative is "(E) none of the others." If you believe none of the first four alternatives is correct, choose response (E).

6. Any scientific or graphing calculator is permitted on Part I. (Unacceptable machines include computers, PDAs, pocket organizers, cell phones, and similar devices. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator.) No other devices are permitted.

7. No one is permitted to explain to you the meaning of any question. Do not ask anyone to violate the rules of the competition. If you have questions concerning the instructions, ask them now.

8. You may now open the test booklet and begin.

1. A shoe store marks up the price of its shoes by 125% over cost. A pair of shoes goes on sale for 20% off. Later, it goes on the clearance rack for an additional 25% off. A customer buys the shoes with a 10% off coupon good for all clearance items. Express the store's profit as a percentage of the original cost.

   A: 67.5% B: 70% C: 21.5% D: −32.5% E: none of the others

2. If 2011 unbiased coins are tossed, what is the probability that there are an even number of heads?

   A: $\frac{2}{3}$ B: $\frac{2^{2011}}{3^{2011}}$ C: $\frac{1}{4}$ D: $\frac{1}{2}$ E: none of the others

3. Let F be the temperature in degrees Fahrenheit and C be the temperature in degrees Celsius. The quantities F and C are related by the formula $F = \frac{9}{5}C + 32$. The Canadian approximation is $F \approx 2C + 30$. When is the Canadian approximation wrong by at most two degrees Fahrenheit?

   A: none of the others B: $32 \le F \le 100$ C: $0 \le F \le 100$ D: $0 \le F \le 32$ E: $38 \le F \le 98$

4. Let $\lfloor x \rfloor$ be the greatest integer that is $\le x$. For example, $\lfloor 3.14 \rfloor = 3$ and $\lfloor 7 \rfloor = 7$. How many integers n satisfy $\lfloor \sqrt{n} \rfloor^2 = n$ and $1 \le n \le 2011$?

   A: 2011 B: 44 C: 0 D: 201 E: none of the others

5. If $\sin\alpha + \cos\alpha = \frac{17}{13}$, then $\sin 2\alpha =$

   A: $\frac{120}{169}$ B: $\frac{60}{169}$ C: $\frac{4}{13}$ D:$-\frac{4}{13}$ E: none of the others

6. Let $a \ge 2$ and $b \ge 2$ be positive integers. Suppose the base a representation of b^2 is 10. What is the base b representation of $a + a^2 + a^3$?

   A: 1010101 B: 1010100 C: 1001 D: 1001001 E: none of the others

7. There are n numbers (not necessarily distinct), each of them is between 13 and 19. Their product is 2940. What is n?

   A: 3 B: 4 C: 5 D: not enough information E: none of the others

8. A 20 volume encyclopedia is arranged in usual order on a bookshelf, in ascending order, with volume 1 on the left. Each volume is 2 inches thick, including the covers, which are each 1/4 inch thick. A worm burrows directly from page 1 of volume 1 to the last page of volume 20. How long a hole must the worm bore?

   A: 40 1/2 inches B: 40 inches C: 39 1/2 inches D: 38 inches
   E: none of the others

9. A quadrilateral with vertices A, B, C, and D (in order) is inscribed in a circle. The angle at A is $120°$, and the angle at B is $90°$. Then the angles at C and D (in order) are:

   A: $40°$ and $90°$ B: $75°$ and $75°$ C: $30°$ and $120°$ D: $60°$ and $80°$
   E: none of the others

10. Suppose that a, b, c are all positive real numbers, and that

$$(\log_b a)(\log_c b)(\log_a x) = 2.$$

   Solve for x.

   A: $x = ac/b^2$ B: $x = c$ C: $x = c^2$ D: $x = 1$ E: none of the others

11. How many five digit integers are multiples of 3 or multiples of 5 (or both)?

   A: 41,999 B: 42,000 C: 42,001 D: 48,000 E: none of the others

12. There are exactly $4! = 24$ distinct 4-digit numbers whose digits are the numbers $1, 2, 3, 4$ in some order. What is the sum of these numbers?

   A: 88888 B: 11110 C: 66666 D: 66660 E: none of the others

13. Two circles, each with radius 1 unit, are placed on a plane so that the distance between their centers is $\sqrt{3}$ units. What is the area of their intersection?

   A: $\pi/3 - \sqrt{3}/2$ B: $2\pi/3 - \sqrt{3}$ C: $2\pi/3 - \sqrt{3}/2$ D: $3\pi/4 - \sqrt{3}/2 - \sqrt{5}$ E: none of the others

14. Alice and Bob play a game, alternately flipping a fair coin. Alice flips first. If Bob's first flip matches Alice's, then Bob wins and the game ends. Otherwise Alice flips again. If Alice's second flip matches Bob's first flip, then Alice wins, and the game ends. Otherwise Bob flips a second time. He wins if his second flip matches Alice's second flip. The continue in this way until somebody wins by matching the previous toss. What is the probability that Alice will win on her third flip?

   A: 1/8 B: 1/16 C: 3/32 D: 1/32 E: none of the others

15. What is the sum of the coefficients of the polynomial after expansion of $(25 - 60x + 22x^2 + 18x^3 - 6x^8)^{2011}$?

   A: 1 B: -1 C: 5262 D: 0 E: none of the others

16. Let $x > 0$. Suppose $x^4 + \frac{1}{x^4} = a$. Find $x + \frac{1}{x}$.

   A: $\sqrt{\sqrt{a+3}+3}$ B: $\sqrt{\sqrt{a+1}+1}$ C: $\sqrt{\sqrt{a+2}+2}$ D: $\sqrt{\sqrt{a+\sqrt{2}}+\sqrt{2}}$ E: none of the others

17. Suppose $a_1 = 1, a_2 = 2$ and $a_n = a_{n-1}/a_{n-2}$ for $n \geq 3$. Find a_{2011}.

   A: $\frac{2}{3}$ B: 2 C: $\frac{1}{2}$ D: 1 E: none of the others

18. Find the number of rational roots of the following polynomial:

$$x^{2011} + \binom{2011}{2009}x^{2009} + \binom{2011}{2007}x^{2007} + \cdots + \binom{2011}{3}x^3 + \binom{2011}{1}x.$$

A: 1 B: 0 C: 2 D: 2011 E: none of the others

19. Let $0 < a < b$. Find the area of the region in the Cartesian plane satisfying $a \leq |x| + |y| \leq b$.
A: $\sqrt{2}b^2 - \sqrt{2}a^2$ B: $b^2 - a^2$ C: $2b^2 - 2a^2$ D: $\frac{1}{2}b^2 - \frac{1}{2}a^2$
E: none of the others

20. Eight tiles can be arranged to spell LADYGAGA. How many distinct 3 letter sequences can be formed from these tiles?

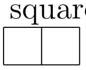
A: 336 B: 85 C: 56 D: 120 E: none of the others

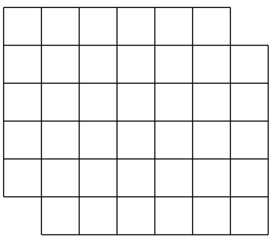
21. For how many integers k does the polynomial

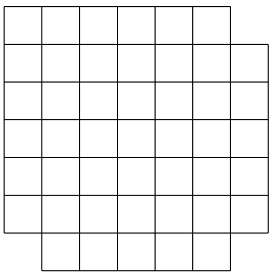
$$x^3 + (k+4)x^2 + (k^2 + k + 3)x + k^2$$

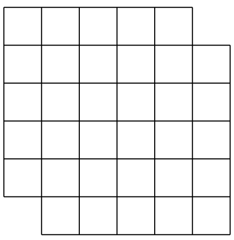
have three distinct real roots?

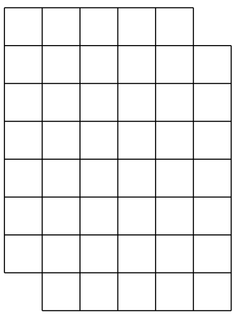
A: 3 B: 2 C: 4 D: 5 E: none of the others

22. Which of the arrays of squares shown below can be covered by tiles of the following shape with no overlapping? 

A: 

B: 

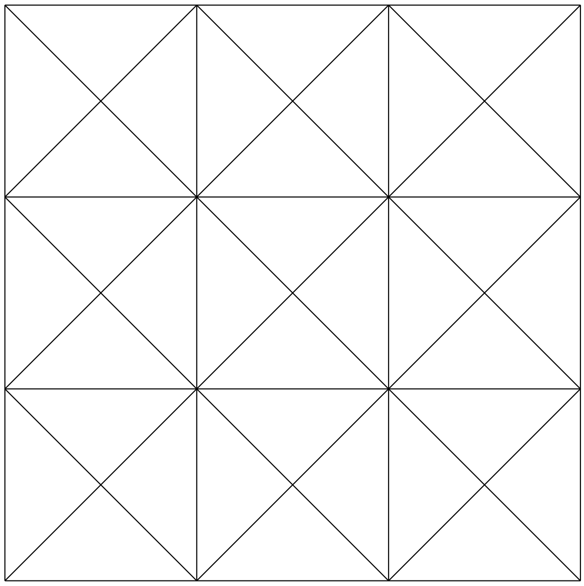
C: 

D: 

E: none of the others

23. How many triangles are contained in the figure below?

A: 72 B: 180 C: 120 D: 96 E: 124



24. A collection of pennies is arranged in a hexagonal array so that each of the interior pennies touches 6 others. There are 10 pennies along each side of the hexagon, counting both corners. How many pennies does the array contain?

A: 331 B: 270 C: 361 D: 271 E: none of the others

25. Ashley, Brandon, Chloe, Daniel, Emily, and Frank each make a statement during the investigation of an incident in the school. Their statements are as follows. Each statement is either true or false.

Ashley:	The math teacher did it.
Brandon:	Ashley's statement is false.
Chloe:	Brandon, Daniel, Emily, and Frank all made false statements.
Daniel:	Both Emily and Frank made true statements.
Emily:	Chloe's statement is true.
Frank:	Either Brandon's statement is true and Daniel's is false, or vice versa.

How many of the statements are true?

A: 2 B: 3 C: 4 D: 5 E: 1

26. For a positive integer n, let $\tau(n)$ denote the number of divisors of n. For example, $\tau(5) = 2$ and $\tau(6) = 4$. Let m be the largest odd divisor of 2011!. Which of the following statements is true?

A: $3 \leq \dfrac{\tau(5)\tau(m)}{\tau(5m)}$ B: $\dfrac{\tau(5)\tau(m)}{\tau(5m)} = 1$ C: $2 \leq \dfrac{\tau(5)\tau(m)}{\tau(5m)} < 3$

D: $1 < \dfrac{\tau(5)\tau(m)}{\tau(5m)} < 2$ E: none of the others

27. For real values of x and y, the minimum value of $f(x,y) = x^2 + 2xy + 2y^2 + 6y + 19$ is

A: 10 B: 0 C: 19 D: $-\infty$ E: none of the others

28. Start with an equilateral triangle of side length 1. Construct a second triangle by connecting the midpoints of the sides of the first triangle. Construct a third triangle by connecting the midpoints of the edges of the second triangle. Continue this process indefinitely. What is the sum of the areas of all of the triangles?

A: $\sqrt{3}$ B: $\frac{2}{\sqrt{3}}$ C: $\frac{1}{\sqrt{3}}$ D: ∞ E: none of the others

29. A standard clock has a minute hand that rotates through 360° in 1 hour, and an hour hand that rotates through 360° in 12 hours. At midnight, the hands are pointing vertically. At what time, to the nearest second, will they next point in the same direction?

A: 1:05:33 AM B: 1:05 AM C: 1:06 AM D: 1:07:26 AM
E: 1:05:27 AM

30. Let $a_1, a_2, \ldots$ be an infinite sequence of real numbers such that

$$a_1 + a_2 + \cdots + a_n = n2^n$$

for all positive integers n. Then for all positive integers n,

A: $a_n = (n+1)2^{n-1}$ B: $a_n = n2^{n/2}$ C: $a_n = n2^{n-1}$ D: $a_n = n2^{n-2}$
E: none of the others

31. Let C be a circle, T_1 an equilateral triangle inscribed in the circle, and T_2 an equilateral triangle with sides tangent to C. Then $(\text{area}\, T_2)/(\text{area}\, T_1) =$

A: 2 B: 4 C: 3 D: $\sqrt{3}$ E: none of the others

32. Given a set of positive integers, you are allowed to introduce new members that are the difference or sum of numbers in the set. For example, if the initial set is $\{2, 5, 8\}$, then you could introduce $8 - 5 = 3$ and $2 + 8 = 10$ to create a new set, $\{2, 3, 5, 8, 10\}$. You are allowed to repeat this process as many times as you like, creating larger and larger sets, but only containing positive numbers. If the initial set is $\{56, 91, 147\}$, then the smallest number that can occur in subsequent sets is

A: 1 B: 7 C: 21 D: 56 E: none of the others

33. For how many integers n does the polynomial $x^3 + nx + 2$ have an integer root?

A: 0 B: 4 C: 3 D: infinitely many E: none of the others

34. The letters a, b, c, d, e, f, g, h represent the numbers $1, 2, 3, 4, 5, 6, 7, 8$, though not necessarily in that order. If $a \cdot b \cdot c = 140$ and $d \cdot e = g - f = 6$, then what is the value of h?

A: $h = 4$ B: $h = 3$ C: $h = 6$
D: The given information is not sufficient to determine the value of h.
E: none of the others

35. Let a be a real number and n a positive integer. What is the remainder when $x^{2n+1} - 1$ is divided by $x^2 - a^2$?

A: $a^{2n+1} + 1$ B: $a^{2n}x + 1$ C: $a^{2n}x - 1$ D: $-a^{2n+1} + 1$
E: none of the others

36. Dan, Hugh, and Eddie are in a bicycle race. As they pass through the town of Sidney, MI, they are all traveling at the velocity 31 km/hr. However, Dan is 2 kilometers ahead of Hugh, and Hugh is $1\frac{1}{2}$ kilometers ahead of Eddie. For the remainder of the race, Dan keeps the same velocity, Hugh accelerates at a constant rate of a km/hr^2, and Eddie accelerates at a constant rate of b km/hr^2. The three riders finish the race at the same time. What is the ratio b/a?

A: $\frac{\sqrt{7}}{2}$ B: $\frac{5}{4}$ C: $\frac{7}{4}$ D: $\frac{\sqrt{31}}{2}$ E: none of the others

37. The cubic polynomial $x^3 + ax^2 + b$ has three roots: $2, 4$, and r. Find r.

A: $r = \frac{1}{8}$ B: $r = \frac{3}{4}$ C: $r = -\frac{4}{3}$ D: $r = -6$ E: none of the others

38. Evaluate $1^2 - 2^2 + 3^2 - 4^2 + \cdots + 2011^2$.

A: 3023066 B: 2032066 C: 2023066 D: 3032066 E: none of the others

39. Three ordinary dice are rolled. What is the probability that the three will show three different faces and that none of the faces shown will be a six?

A: 5/18 B: 1/3 C: 125/216 D: 5/9 E: none of the others

40. There are n even numbers and n odd numbers in a set. Three numbers are picked at random from the set (without replacement). Suppose the probability that the three numbers have the same parity is 2/9. What is n?

A: 11 B: 14 C: 15 D: 16 E: none of the others

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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