FORTY-NINTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part I

October 5, 2005

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Carefully read and follow the instructions printed on the answer sheet. Check to ensure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor asks you to stop, please quit working immediately and turn in your answer sheet.

3. Consider the problems and responses carefully. You may work out ideas on scratch paper before selecting a response.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.

5. For each of the questions, five different possible responses are provided. In some cases the fifth alternative is "(e) none of the above." If you believe none of the first four alternatives is correct, choose response (e).

6. Any scientific or graphing calculator is permitted on Part I. Unacceptable machines include computers, PDAs, pocket organizers and similar devices. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator.

7. No one is permitted to explain to you the meaning of any question. Do not ask anyone to violate the rules of the competition. If you have questions concerning the instructions, ask them now.

8. You may now open the test booklet and begin.

1. For every triple (a, b, c) of non-zero real numbers, form the number

$$\frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} + \frac{abc}{|abc|}.$$

The set of real numbers formed is

(a) $\{0\}$ (b) $\{-2, 0, 2\}$ (c) * $\{-4, 0, 4\}$ (d) $\{-4, -2, 2, 4\}$
(e) $\{-4, -2, 0, 2, 4\}$

2. Simplify $\left[\sqrt[3]{\sqrt[6]{a^9}}\right]^4 \left[\sqrt[6]{\sqrt[3]{a^9}}\right]^4$ where a is a positive real number.

(a) a^2 (b) $*a^4$ (c) a^8 (d) a^{12} (e) a^{16}

3. Suppose k is a real number such that $kx^2 + (k+1)x + 3 = 0$ has exactly one real root. Which of the following could be a value for k?

(a) $-5 + 2\sqrt{5}$ (b) $-5 + 2\sqrt{6}$ (c) $5 - 2\sqrt{5}$ (d) $5 + 2\sqrt{5}$ (e) $*5 + 2\sqrt{6}$

4. Let $f^{-1}(x)$ be the inverse function of $f(x) = 2x + 1$. Let $g(x) = 6x^2 + 4x + 3$. Then $(f^{-1} \circ g)(x) =$

(a) $\dfrac{1}{12x^2 + 8x + 7}$ (b) $\dfrac{6x^2 + 4x + 3}{2x + 1}$ (c) $*3x^2 + 2x + 1$
(d) $12x^3 + 14x^2 + 10x + 3$ (e) none of the above.

5. The operation $*$ is defined by $a * b = \dfrac{3a - 2b}{2ab}$ for $a, b \neq 0$. If $x * y = \frac{1}{4}$ and $y * x = -1$, then $x * x$ is equal to

(a) -1 (b) $*-\dfrac{1}{2}$ (c) $-\dfrac{1}{4}$ (d) $\dfrac{1}{2}$ (e) 1

6. The graphs of $y = \dfrac{x^2 - 4}{x - 2}$ and $y = 2x$ intersect in

(a) *no points (b) one point whose x-coordinate is 0
(c) one point whose x-coordinate is 2
(d) two distinct points (e) more than two points

7. In how many ways can 7 people be seated in a row if two of them, Eddie and John, refuse to sit next to each other?

(a) 720 (b) *3600 (c) 4320 (d) 5039 (e) 5040

8. If an angle of a triangle remains unchanged but each of its two including sides is doubled, then the new area divided by the old area is

(a) 2 (b) 3 (c) *4 (d) 5 (e) more than 6

9. If a, b, and c are positive integers, then $\sqrt{a+\frac{b}{c}}$ and $a\sqrt{\frac{b}{c}}$ are equal if and only if

(a) $*c = \frac{b(a^2-1)}{a}$ (b) $a = b = c$ (c) $a = b$ and $c = a - 1$
(d) $a = b$ and $c = a = 1$ (e) $a = b$ and c is any value

10. The mirror image of the point $(0, 1)$ reflected across $2x + 3y = 6$ is

(a) $(1, 0)$ (b) $(1, 3)$ (c) $(1.5, 2.5)$ (d) $(1.5, 3.0)$ (e) *none of the above

11. A survey of 100 high school students revealed the following data: 28 eat chicken, 30 eat beef, 20 eat pork. These numbers include 12 students who eat chicken and beef, 7 who eat chicken and pork, 5 who eat beef and pork, and 3 who eat all three. How many non-meat eaters are in the sample?

(e) 39 (b) 40 (c) 41 (d) 42 (a) *43

12. Given that $\frac{5}{2} = \sqrt{a} + \frac{1}{\sqrt{a}}$ where $0 < a < \frac{1}{2}$, $a - \frac{1}{a}$ is equal to

(a) $*-\frac{15}{4}$ (b) $-\frac{3}{2}$ (c) $\frac{3}{2}$ (d) $\frac{15}{4}$ (e) none of the above

13. Suppose that the function f is defined on the set of integers by $f(n) = \begin{cases} n^2 - n & \text{if } n \text{ is odd} \\ n - 1 & \text{if } n \text{ is even} \end{cases}$. If $f(f(f(n))) = 109$ for a positive integer n, then the product of the digits of n is

(a) 0 (b) 1 (c) *2 (d) 3 (e) 30

14. How many three-digit numbers contain the digits 1 and 3 but none of the digits 0, 2, or 5?

(a) 15 (b) 21 (c) 30 (d) *36 (e) 42

15. For a positive integer n, let $E_n = 2+4+6+\cdots+2(n+1)$, that is, E_n is the sum of the first $n+1$ even positive integers. Let S be the set of all positive integer n for which E_n is a positive power of 2 (a positive power of 2 is an integer of the form 2^m where m is a positive integer). Which of the following statement is correct?

(a) $*S$ is the empty set.
(b) S contains exactly one positive integer.
(c) S contains exactly two positive integers.
(d) S contains more than two, but finitely many, positive integers.
(e) S contains infinitely many positive integers.

16. The area of the triangle in the xy-plane bounded by $x+y=5, 2x-y=4$, and the y-axis is

(a) 7.5 (b) 13 (c) $*13.5$ (d) 14 (e) 27

17. If x workers take y days to build z houses, how many days would q workers take to build r houses?

(a) $\frac{qry}{xz}$ (b) $*\frac{xyr}{qz}$ (c) $\frac{qz}{rxy}$ (d) $\frac{ryz}{qx}$ (e) none of the above

18. The largest possible area of a rectangle inscribed in the ellipse $\frac{x^2}{1800}+\frac{y^2}{200}=1$ is

(a) 800 (b) 1000 (c) 1120 (d) $*1200$ (e) 1280

19. If a, b, and c are the roots of $2x^3+4x^2+px+q=0$ then $a+b+c=$

(a) -4 (b) $*-2$ (c) 2 (d) 4 (e) none of the above

20. How many solutions does $\cos(x+\frac{\pi}{6})=0.5,\ 0\le x\le 1000$ have?

(a) 1 (b) 159 (c) 160 (d) 316 (e) $*319$

21. A dolphin travels m feet due north at 2 minutes per mile. The dolphin returns due south to the starting point at 1/3 mile per minute. The average speed in miles per hour for the entire trip is

(a) 6 (b) 12 (c) $*24$ (d) 36 (e) 48

22. The number of pairs of positive integers (x, y) that satisfy the equation
$x^2+y^2=x^3$ is

(a) 0 (b) 1 (c) 2 (d) ∗not finite (e) none of the above

23. If $2^x = \cos(y/2)$ and $a^x = \sin y$, then $\sin(y/2)$ is equal to

(a) $*\frac{1}{2}\left(\frac{a}{2}\right)^x$ (b) $\left(\frac{a}{2}\right)^x$ (c) $\frac{a^x}{2}$ (d) $2^{\frac{x}{2}}$ (e) $a^x - 2^x$

24. Let ABC be a triangle with $AB = a$, $BC = a + b$, and $AC = a + 2b$, $b > 0$. Which of the following must be true?

I. $a > b$ II. $\angle C < 60°$ III. $a < \frac{(1+\sqrt{3})}{2} b$

(a) I only (b) II only (c) III only (d) ∗I and II only
(e) II and III only

25. What is the set of values of k for which the function $f(x) = \cos^4 x - k^2 \cos^2(2x) + \sin^4 x$ is constant?

(a) $*\left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$ (b) $\left\{\frac{1}{2}, -\frac{1}{2}\right\}$ (c) $\left\{\frac{1}{2}\right\}$ (d) $\left\{\frac{1}{\sqrt{2}}\right\}$
(e) cannot be determined from the given information

26. Given that $ab - ac = 0$, $xb - xd = 0$ and $c < d$. Which of the following must be true?

(a) $x = 0$ (b) $*ax = 0$ (c) $a = 0$ (d) $b = c$ (e) none of the above

27. How many real number solutions does $(\frac{x-21}{2x})^{x^2-x-12} = 1$ have?

(a) 0 (b) 1 (c) 2 (d) 3
(e) ∗4

28. Let S be the set of all ordered triples of real numbers (x, y, z) that satisfy the equation

$$3x^2 + 6y^2 + z^2 + 4xy - 2yz = 0.$$

Which of the following is correct?

(a) S is the empty set.
(b) ∗S contains exactly one ordered triple.
(c) S contains exactly two ordered triples.
(d) S contains more than two, but finitely many, ordered triples.
(e) S contains infinitely many ordered triples.

29. If $\sin\theta + \cos\theta = 1/5$ and $0 \le \theta < \pi$, then $\tan\theta$ is

(a) $*-\frac{4}{3}$ (b) $-\frac{3}{4}$ (c) $\frac{3}{4}$ (d) $\frac{4}{3}$
(e) not completely determined by the given information

30. If p, q, and M are positive integers and $q < 100$, then the number obtained by increasing M by $p\%$ and decreasing the result by $q\%$ exceeds M if and only if

(a) $p > q$ (b) $p > \frac{q}{1-q}$ (c) $p > \frac{100q}{100+q}$ (d) $p > \frac{q}{100-q}$ (e) $*p > \frac{100q}{100-q}$

31. Let ABC be a triangle. Suppose $\angle A = 60°$ and the bisectors of $\angle B$ and $\angle C$ meet $\overline{AC}$ and $\overline{AB}$ at points P and Q, respectively. Let M be a point of intersection of $\overline{BP}$ and $\overline{CQ}$. Which of the following are true?

I. $\angle BMC = 120°$
II. $\angle MPQ = 30°$
III. $MP = MQ$

(a) I only (b) II only (c) III only (d) I and II only (e) *I, II and III

32. A circle is inscribed in the right triangle ΔABC with hypotenuse $\overline{AB}$. If $\overline{AC}$ is tangent to the circle at D such that $AD = 4$ and $DC = 3$, then the area of ΔABC is

(a) 70 (b) 77 (c) 80.5 (d) *84 (e) 91

33. If $v = gt + v_0$ and $s = \frac{1}{2}gt^2 + v_0t$ then express t in terms of s, v, and v_0.

(a) $*\frac{2s}{v+v_0}$ (b) $\frac{2s}{v_0-v}$ (c) $\frac{2s}{v-v_0}$ (d) $\frac{2s}{v}$ (e) $2s - v$

34. Let $ABCD$ be a rectangle and let P and Q be points on $\overline{BC}$ and $\overline{CD}$, respectively, such that $AP = 8$, $PQ = 6$, $AQ = 10$, and $\angle BAP = 45°$. The value $\cos(\angle DAQ)$ is

(a) $\frac{\sqrt{2}}{2}$ (b) $\frac{\sqrt{2}}{10}$ (c) $*\frac{7\sqrt{2}}{10}$ (d) $\frac{3}{5}$ (e) none of the above

35. The perimeter of a semicircular region, measured in meters, is numerically equal to its area, measured in square meters. The radius of the semicircle, measured in meters, is

(a) $\frac{2}{\pi}$ (b) $\frac{1}{2}$ (c) π (d) 1 (e) $*\frac{4}{\pi} + 2$

36. The remainder of $x^{5000} + x^{4999} + \cdots + x + 1$ divided by $x + 1$ is

(a) -1 (b) $*1$ (c) 4999 (d) 5000 (e) 5001

37. The angle between the hour and minute hands of an ordinary clock showing 6 : 45 is

(a) 67° (b) *67.5° (c) 68° (d) 68.5° (e) 70°

38. Which of the following is equal to

$$\frac{1}{\log_2 100!} + \frac{1}{\log_3 100!} + \cdots + \frac{1}{\log_{100} 100!}?$$

(a) $\log_{100!} 5049$ (b) $\frac{100}{\log_{100} 100!}$ (c) 0 (d) $*1$ (e) none of the above

39. A fair die is rolled six times. The probability of rolling more than 4 at least five times is

(a) $\frac{2}{729}$ (b) $\frac{3}{729}$ (c) $\frac{12}{729}$ (d) $*\frac{13}{729}$ (e) none of the above

40. What day of the week will it be 2005^{2005} days after a Monday?

(a) Tuesday (b) Wednesday (c) *Thursday (d) Friday (e) Saturday

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

DIRECTOR
Eddie Cheng
Oakland University

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