FORTY-EIGHTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part I

October 6, 2004

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Carefully read and follow the instructions printed on the answer sheet. Check to ensure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.
2. Do as many problems as you can in the 100 minutes allowed. When the proctor asks you to stop, please quit working immediately and turn in your answer sheet.
3. Consider the problems and responses carefully. You may work out ideas on scratch paper before selecting a response.
4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.
5. For each of the questions, five different possible responses are provided. In some cases the fifth alternative is "(e) None of the above." If you believe none of the first four alternatives is correct, choose response (e).
6. Any scientific or graphing calculator is permitted on Part I. Unacceptable machines include computers, PDAs, pocket organizers and similar devices. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator.
7. No one is permitted to explain to you the meaning of any question. Do not ask anyone to violate the rules of the competition. If you have questions concerning the instructions, ask them now.
8. You may now open the test booklet and begin.

1. If $f(x) = x^2 + 4x$, $a = 1.01$, and $b = 1$, then the value of $\dfrac{f(b) - f(a)}{b - a}$ is

(a) 6 (b) 6.01 (c) 6.02 (d) 6.03 (e) None of the above.

2. If $a/b = 3/4$, $b/c = 8/9$ and $c/d = 2/3$, then ad/b^2 is equal to

(a) $\dfrac{81}{64}$ (b) $\dfrac{64}{81}$ (c) $\dfrac{61}{84}$ (d) $\dfrac{84}{61}$ (e) 1

3. The coefficient of x^9 when $\left(x + \dfrac{2}{\sqrt{x}}\right)^{30}$ is expanded and simplified is

(a) $\binom{30}{9} 2^9$ (b) $\binom{30}{9} 2^{21}$ (c) $\binom{30}{16} 2^{14}$ (d) $\binom{30}{9}$ (e) None of the above.

4. If you begin with a circle of radius 1, inscribe an equilateral triangle in the circle, inscribe a circle in the triangle, inscribe an equilateral triangle in the smaller circle, and repeat this process indefinitely, then the sum of the areas of all of the circles is

(a) 2π (b) $\dfrac{5\pi}{3}$ (c) $\dfrac{3\pi}{2}$ (d) $\dfrac{4\pi}{3}$ (e) None of the above.

5. If $x = 1$ and $x = 2$ are solutions of the equation $x^3 + ax^2 + bx + c = 0$ and $a + b = 1$, then what does b equal?

(a) 2 (b) 4 (c) 5 (d) 3 (e) None of the above.

6. How many integers between 200 and 700 consist of three distinct digits?

(a) 350 (b) 360 (c) 365 (d) 370 (e) None of the above.

7. The sum of the solutions of the equation $9^x - 6 \cdot 3^x + 8 = 0$ is

(a) $\log_3 2$ (b) $\log_3 6$ (c) $\log_3 8$ (d) $\log_3 4$ (e) None of the above.

8. How many diagonals does a regular polygon with 20 sides have?

(a) 170 (b) 180 (c) 190 (d) 200 (e) None of the above.

9. Let $\triangle ABC$ be the right triangle with vertices of $A(0, 2)$, $B(1, 0)$, and $C(0, 0)$. If D is the point on $\overline{AB}$ such that the segment $\overline{CD}$ bisects angle C, then the length of $\overline{CD}$ is

(a) $\dfrac{1}{\sqrt{2}}$ (b) $\dfrac{\sqrt{5}}{2}$ (c) $\dfrac{\sqrt{3}}{2}$ (d) $\dfrac{2\sqrt{2}}{3}$ (e) None of the above.

10. Let $a > 1$ be a real number and $f(x) = \log_a x^2$ for $x > 0$. If f^{-1} is the inverse function of f and b and c are real numbers, then $f^{-1}(b+c)$ is equal to

(a) $\dfrac{1}{f(b+c)}$

(b) $f^{-1}(b) + f^{-1}(c)$

(c) $f^{-1}(b)f^{-1}(c)$

(d) $\dfrac{1}{f^{-1}(b) + f^{-1}(c)}$

(e) None of the above.

11. When five ordinary six-sided dice are tossed simultaneously, the probability that a 1 shows on the top face of exactly two of the dice is

(a) $\dfrac{625}{3888}$ (b) $\dfrac{1}{3888}$ (c) $\dfrac{2}{5}$ (d) $\dfrac{5}{1944}$ (e) None of the above.

12. The area of the intersection of the planar regions $y^2 \geq 3x^2$ and $x^2 + y^2 \leq 9$ is

(a) 2π (b) 3π (c) 4π (d) $\dfrac{3\pi}{2}$ (e) None of the above.

13. A car driving at a constant rate travels $d/6$ feet in t seconds. If this rate is maintained for 3 minutes, how many yards does the car travel in 3 minutes?

(a) $\dfrac{d}{1080t}$ (b) $\dfrac{30d}{t}$ (c) $\dfrac{30t}{d}$ (d) $\dfrac{10t}{d}$ (e) $\dfrac{10d}{t}$

14. The prime number 1999 can be written as $a^2 - b^2$ where a and b are integers. What does $a^2 + b^2$ equal?

(a) 1,999,000 (b) 1,998,001 (c) 1,997,999 (d) 1,998,000

(e) There is not enough information provided answer the question.

15. A box contains ten balls numbered from 1 to 10. If you pick two balls at random, what is the probability that the sum of the numbers on the two balls is even?

(a) $\dfrac{1}{2}$ (b) $\dfrac{47}{99}$ (c) $\dfrac{45}{99}$ (d) $\dfrac{4}{9}$ (e) $\dfrac{2}{10}$

16. A store sells red apples at four for a dollar and green apples at three for a dollar. Another store sells red apples at four for a dollar and green apples at six for a dollar. You spend \$20 and receive m red apples and n green apples from each store. How many apples do you have?

(a) 100 (b) 80 (c) 60 (d) 40

(e) There is not enough information provided to answer the question.

17. John has taken n tests. If he receives 71 on the next test, his average will be exactly 83. If he receives 99, then the average will be exactly 87. What does n equal?

(a) 6 (b) 5 (c) 8 (d) 7

(e) There is not enough information provided to answer the question.

18. If $(-1, 12)$, $(0, 5)$, and $(2, -3)$ are points on the graph of the equation $y = ax^2 + bx + c$, then what does $a + b - c$ equal?

(a) -5 (b) 5 (c) -10 (d) 10

(e) There is not enough information provided to answer the question.

19. Arrange the following numbers in increasing order: $2^{5555}, 3^{3333}, 6^{2222}$.

(a) $2^{5555}, 3^{3333}, 6^{2222}$
(b) $3^{3333}, 2^{5555}, 6^{2222}$
(c) $2^{5555}, 6^{2222}, 3^{3333}$
(d) $6^{2222}, 2^{5555}, 3^{3333}$
(e) $3^{3333}, 6^{2222}, 2^{5555}$

20. If $x + y = 1$ and $x^2 + y^2 = 4$, then what does $x^3 + y^3$ equal?

(a) $\frac{11}{2}$ or $\frac{5}{2}$ or 10 (b) $\frac{11}{2}$ or $\frac{5}{2}$ (c) 10 (d) $\frac{5}{2}$ (e) $\frac{11}{2}$

21. Find the number of triangles with sides of integer length whose perimeter is 10.

(a) 4 (b) 3 (c) 2 (d) 1 (e) 0

22. If $f(x) = \cos(x)$ and $g(x) = \sin^{-1}(x)$, then $(f \circ g)(x)$ is equal to

(a) $x^2 - 1$ (b) $1 - x^2$ (c) $\sqrt{x^2 - 1}$ (d) $\sqrt{1 - x^2}$ (e) None of the above.

23. Let $ABCD$ be a quadrilateral with side $\overline{AB}$ extended to E so that $\overline{AB} = \overline{BE}$. Segments $\overline{AC}$ and $\overline{CE}$ are drawn to form angle $\angle ACE$. If $\angle ACE$ is a right angle which of the following must be true about $ABCD$?

(a) All of the sides are equal.
(b) All of the angles are equal.
(c) $ABCD$ is a parallelogram.
(d) At least two angles are equal.
(e) At least two sides are equal.

24. A square of side length is r has a square inside it with one-half of the area of the larger square. There is a uniform border between the two squares. Find the width of the border.

(a) $\dfrac{r(2+\sqrt{2})}{4}$

(b) $\dfrac{r(2-\sqrt{2})}{4}$

(c) $\dfrac{r(2+\sqrt{2})}{4}$ and $\dfrac{r(2-\sqrt{2})}{4}$

(d) $\dfrac{r(\sqrt{2}-2)}{4}$ and $\dfrac{r(\sqrt{2}+2)}{4}$

(e) None of the above.

25. Let $a > 1$ be a real number. If S is the set of real numbers x that are solutions to the equation $a^{2\log_2 x} = 5 + 4x^{\log_2 a}$, then

(a) S is the empty set.

(b) S contains infinitely many real numbers.

(c) S contains exactly two real numbers.

(d) S contains more than two, but finitely many, real numbers.

(e) S contains exactly one real number.

26. Solve the inequality $\dfrac{2}{x+1} < \dfrac{1}{x-1}$.

(a) $1 < x < 3$

(b) $-3 < x < -1$

(c) $x < -1$ or $x > 1$

(d) $x < -1$ or $1 < x < 3$

(e) $-3 < x < -1$ or $x > 1$

27. The set of values of k for which the sum of the roots of $P(x) = 4x^2 + k^2x + k$ is equal to twice the product of the roots is

(a) $\{-2\}$ (b) $\{-2, 0\}$ (c) $\{0\}$ (d) $\{2\}$ (e) None of the above.

28. In how many ways can six boys and five girls stand in a row if all the girls are to stand together but the boys cannot all stand together?

(a) 172,800 (b) 30 (c) 432,000 (d) 86,400 (e) None of the above.

29. How many subsets of the set $\{0, 1, 2, \ldots, 10\}$ contain either 2, 4, or 6?

(a) 256 (b) 896 (c) 1,536 (d) 1,792 (e) None of the above.

30. If $x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}$, then the exact value of x is

(a) $-1 + \sqrt{2}$ (b) $\frac{1}{2} + \sqrt{2}$ (c) $\frac{1}{4}$ (d) $\sqrt{2}$ (e) None of the above.

31. If a and b are positive real numbers such that $b > a$, and c and d are negative real numbers such that $d > c$, then which of the following statements must be true?

I. $\frac{b}{d} > \frac{a}{c}$
II. $b - d > a - c$
III. $(a + d)(b + c) > (a + c)(b + d)$

(a) only I. (b) only II. (c) only III. (d) only I and II. (e) I, II, and III.

32. What does the sum $\frac{1}{1+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \cdots + \frac{1}{\sqrt{2n-1}+\sqrt{2n+1}}$ equal?

(a) $\frac{-1 + \sqrt{2n+1}}{2}$
(b) $\frac{1 + \sqrt{2n+1}}{2}$
(c) $\frac{1}{1 + \sqrt{2n+1}}$
(d) $\frac{n}{1 + \sqrt{2n-1}}$
(e) None of the above.

33. If two real numbers x and y satisfy the equation $\frac{x}{y} = x - y$, then which of the following is true?

(a) Both x and y must be rational.
(b) It is possible that $y = 1$.
(c) Both x and y must be irrational.
(d) It is not possible that x and y are both integers.
(e) $x \geq 4$ or $x \leq 0$.

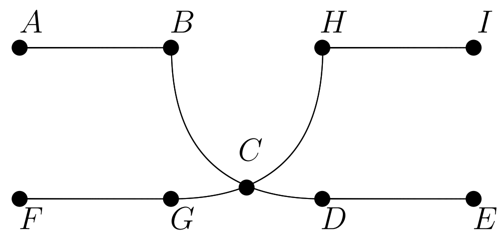
34. Suppose that a, b, c, and d are real numbers such that $ad - bc \neq 0$ and $f(x) = \frac{ax + b}{cx + d}$. If f^{-1} is the inverse function of f, then $f^{-1}(x)$ is equal to

(a) $\frac{cx - a}{dx - b}$ (b) $\frac{cx + d}{ax + b}$ (c) $\frac{bx - d}{cx - b}$ (d) $\frac{dx - b}{-cx + a}$ (e) $\frac{c + dx}{a + bx}$

35. If $S = i^n + i^{-n}$, where $i = \sqrt{-1}$ and n is an integer, how many distinct values are possible for S?

(a) 1 (b) 2 (c) 3 (d) 4 (e) more than 4.

36. The following figure shows two intersecting roads: $ABDE$ and $FGHI$. The points F, G, D, and E lie on one line. Segments $\overline{AB}$ and $\overline{FG}$, and $\overline{HI}$ and $\overline{DE}$ of the roads are parallel and they are 20 meters apart. Arcs BD and GH are quarter circle arcs intersecting at C, constructed using B and H as centers. Find the distance from C to the line through F, G, D, and E



(a) $10(2-\sqrt{3})$ meters
(b) $10(2-\sqrt{2})$ meters
(c) $20(\sqrt{2}-1)$ meters
(d) $10(\sqrt{2}-1)$ meters
(e) None of the above.

37. Which of the following statements is false?

(a) If $\gcd(n, 3) = 1$ then $n^2 - 1$ is a multiple of 3.
(b) For any odd integer n, $n^2 - 1$ is a multiple of 4.
(c) For any odd integer n, $n^2 - 1$ is a multiple of 8.
(d) For any even integer n, $n^2 - 1$ is not a multiple of 2.
(e) For any even integer n, $n^2 - 1$ is a multiple of 3.

38. Five straight lines are drawn in a plane with no two lines parallel and no three lines concurrent. Find the number of regions into which they divide the plane. (Three lines are called *concurrent* if they intersect in the same point.)

(a) 8 (b) 10 (c) 16 (d) 20 (e) 22

39. If $2^x = 8^{y+1}$ and $9^y = 3^{x-9}$, then what is the value of $x + y$?

(a) 30 (b) 27 (c) 24 (d) 21 (e) 18

40. Mr. and Mrs. Olde invite all their descendents and the spouses of their descendents to a party. They are all present. The party has 85 people not including Mr. and Mrs. Olde. Suppose each of their children and grandchildren is married but none of their great-grandchildren. They have no great-great-grandchildren. If they have three times as many grandchildren as children and three times as many great-grandchildren as grandchildren. How many children do they have?

(a) 4 (b) 6 (c) 2 (d) 5 (e) 10

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

DIRECTOR
B. David Redman, Jr.
Delta College

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