# FORTY-FIFTH ANNUAL
# MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

## Part I

October 10, 2001

## INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. Check to ensure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please quit working immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.

5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "e) none of these". If you believe none of the first four alternatives to be correct, mark e) in such cases.

6. Any scientific or graphing calculator is permitted on Part I. Unacceptable machines include portable computers and pocket organizers. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator.

7. No one is permitted to explain to you the meaning of any question. Do not request anyone to break the rules of the competition. If you have questions concerning the instructions, ask them now.

8. You may now open the test booklet and begin.

1. A line in the plane is parallel to the line $2x + 7y = 3$ and passes through the point $(3, -1)$. What is $y$-intercept of this line?

   a) $-\frac{3}{7}$ b) $-\frac{1}{7}$ c) $\frac{1}{7}$ d) $\frac{3}{7}$ e) The line never intersects the $y$-axis.

2. There are 24 students in a math class. Ten of them are male and of these, four are left-handed. If eleven of these students are right-handed females, how many left-handed students are in the class?

   a) 3 b) 5 c) 6 d) 7 e) 8

3. A quadratic polynomial $P(x) = x^2 + bx + c$ satisfies $P(3) = -2$ and $P(5) = 10$. What is $P(2)$?

   a) $-8$ b) $-5$ c) 5 d) 8 e) 12

4. A woman who is 5 feet, 4 inches tall stands on the shore of Lake Superior. Which of the following best approximates the distance the woman can see out to the horizon? Note: the radius of the earth is approximately 4000 miles, and there are 5280 feet in a mile.

   a) 2 miles b) 3 miles c) 7 miles d) 20 miles e) 50 miles

5. A group of four people consists of two cat lovers and two cat haters. Three of these people are chosen at random. What is the probability that both cat lovers are selected?

   a) 0 b) $\frac{1}{4}$ c) $\frac{1}{3}$ d) $\frac{1}{2}$ e) $\frac{2}{3}$

6. Determine all values of $x$ so that $\frac{1}{x-1} > \frac{1}{x}$.

   a) $x < 0$ b) $0 < x$ c) $0 < x < 1$ d) $1 < x$ e) $x < 0$ or $1 < x$

7. Three circles of radius 1 are placed so that each of them is tangent to the other two. What is the radius of the smallest circle that contains the three circles?

   a) $1 + \frac{1}{\sqrt{3}}$ b) $1 + \frac{1}{\sqrt{2}}$ c) $1 + \frac{2}{\sqrt{3}}$ d) $2 + \frac{1}{\sqrt{3}}$ e) $2 + \frac{1}{\sqrt{2}}$

8. How many real solutions does the equation $\frac{1}{x} = \frac{1}{x+a} + \frac{1}{x-a}$ have if $a \neq 0$?

   a) 0 b) 1 c) 2 d) 3 e) 4

9. The batting average of a baseball player is computed by dividing the number of hits by the number of times at bat. A player has a batting average of .300 after 200 times at bat. How many hits will be needed during the remainder of the season if the player is to finish the season with a batting average of .320 after 500 times at bat?

a) 90 b) 95 c) 100 d) 110 e) 115

10. The Baseball Encyclopedia reports that Rene Gonzales had a better batting average than Mike Fielder in 1987 (Gonzales .267 versus Fielder .266) and again in 1988 (Gonzales .215 versus Fielder .173). However, over the combined two-year period 1987/88, Fielder had a better batting average than Gonzales (Fielder .245 versus Gonzales .226). Which statement most accurately reflects what can be deduced from this information:

a) There is an error in the Baseball Encyclopedia.
b) Gonzales and Fielder must have had the same number of at-bats in 1987 and the same number of at-bats in 1988.
c) Gonzales and Fielder may have had the same number of at-bats in 1987 and the same number of at-bats in 1988.
d) Gonzales and Fielder must have had different numbers of at-bats in 1987 and different numbers of at-bats in 1988.
e) Gonzales and Fielder must have had different numbers of at-bats in at least one of 1987 and 1988.

11. To number all the pages of a book, the printer uses 651 digits. How many pages does the book have?

a) 154 b) 189 c) 217 d) 253 e) 254

12. In the game of Monopoly, there are two ordinary dice, and if you throw doubles (for example, four-four or six-six) three times in a row, you go to jail. What is the probability of rolling three doubles in three consecutive rolls of the dice?

a) $\frac{1}{256}$ b) $\frac{1}{216}$ c) $\frac{1}{36}$ d) $\frac{1}{18}$ e) $\frac{1}{2}$

13. Find all pairs of real numbers $(x, y)$ such that $1, x, y$ forms an arithmetic progression and such that $1 + x^2 = y^2$.

a) $(0, -1)$ and $\left(\frac{4}{3}, \frac{5}{3}\right)$ b) $(0, 1)$ and $\left(\frac{4}{3}, \frac{5}{3}\right)$ c) $(0, -1)$ d) $(0, 1)$ e) $\left(\frac{4}{3}, \frac{5}{3}\right)$

14. The distance from one foot of a parabolic arch to the other is 200 meters. The arch rises to a maximum height of 55 meters above the point midway between the two feet. What is the height of the arch 60 meters along the ground from one foot of the arch to the other foot?

a) 35.2 meters b) 46.2 meters c) 68.0 meters d) 85.8 meters e) 432.1 meters

15. You have two fair eight-sided dice. The first die is labeled $0, 1, 2, 3, 4, 5, 6, 7$, and the second die is labeled $1, 2, 3, 4, 5, 6, 7, 8$. If you throw the dice once and record the sum of the labels on the sides that are facing up, which of the following is most likely to be the sum?

a) 6 b) 7 c) 8 d) 9 e) 10

16. For any positive integer $n$, the function $\tau(n)$ is defined to be the number of factors of $n$. For example, $\tau(10) = 4$, because 10 has 4 factors: 1, 2, 5, and 10. Suppose that $\tau(n)$ is an odd number. What must be true?

a) $n$ is odd b) $n$ is a power of 3 c) $n$ is prime
d) $n$ is a perfect square e) $\tau(n)$ is never odd

17. On a construction site, there is a large number of 10 cm diameter pipes. They were stacked, starting by making a row of pipes side by side, with each pipe in contact with the next on a level surface. A second row was placed on this first row so as to fit into the hollows between the adjacent pipes. This process continues until there are five rows. What is the total height of the pile of pipes, to two decimal places?

a) 33.64 cm b) 43.30 cm c) 44.64 cm d) 47.63 cm e) none of these

18. If $x + y = 1$ and $x^2 + y^2 = 2$, then what is the value of $x^3 + y^3$?

a) $-\frac{1}{2}$ b) 2 c) $\frac{1+\sqrt{3}}{2}$ d) $\frac{5}{2}$ e) none of these

19. The graph of the equation $y + |y| = x + |x|$, restricted to $x < 0$, is

a) the empty set b) a line c) shaped like a V
d) a quadrant of the plane e) two quadrants of the plane

20. We wish to find all circles for which the sum of the area and circumference is $m$. Determine all values of $m$ for which there are exactly two such circles.

a) $m = -\pi$ b) $m > -\pi$ c) $m > 0$ d) $m > \pi$ e) none of these

21. Suppose all of the possible arrangements of 5 letters $Q, R, S, T, U$ are placed in alphabetical order, beginning with $QRSTU$ and ending with $UTSRQ$. If these words are numbered 1 to 120, what would be the 97th word?

a) $TQRSU$ b) $TUSRQ$ c) $TUSQR$ d) $UQRST$ e) $UQRTS$

22. Given a function $f(x)$, let us denote $f(f(a))$ by $f^2(a)$, $f(f(f(a)))$ by $f^3(a)$, and so on. Let $f(x) = -x^3$. Evaluate $f^{1000}(2)$.

a) $-2^{3^{1000}}$ b) $-2^{3000}$ c) $2^{3000}$ d) $3^{2^{1000}}$ e) $2^{3^{1000}}$

23. How many prime numbers exist between 100 and 120?

a) 4 b) 5 c) 6 d) 7 e) 8

24. A Ferris wheel has 6 evenly spaced chairs. The distance measured along a straight line between two neighboring chairs is 12 feet. Which of the following answers is the best approximation to the diameter of the Ferris wheel?

a) 6 feet b) 10.14 feet c) 12 feet d) 15 feet e) 24 feet

25. An urn contains 5 red balls, 4 blue balls, and 3 green balls. Three balls are drawn at random from the urn (one at a time without replacement). What is the probability of having one ball of each color?

a) $\frac{1}{22}$ b) $\frac{3}{22}$ c) $\frac{3}{11}$ d) $\frac{5}{11}$ e) $\frac{5}{24}$

26. Sand is leaking from a giant sand box at a constant rate of $\pi$ cubic inches per minute. The sand forms a cone-shaped pile on the floor beneath the sand box. As it grows, the height of the cone is always half the radius. If the cone is 3 inches tall now, how tall will it be in one hour? Note: The volume of a cone is $\frac{1}{3}\pi r^2 h$ where $h$ is the altitude of the cone and $r$ is the radius of its base.

a) $\sqrt[3]{72}$ inches b) $\sqrt{72}$ inches c) 72 inches d) $\sqrt[3]{72}\,\pi$ inches e) $\sqrt{72}\,\pi$ inches

27. Five identical red balls and two identical blue balls are to be arranged in a row from left to right. In how many ways can they be arranged so that the blue balls are not next to each other?

a) 9 b) 15 c) 30 d) 45 e) 50

28. Let $a$ be a positive integer. Then $\sqrt{a+\sqrt{a}}$ is a root of

a) $x^3 - 2ax^2 + a^2 - a = 0$
b) $x^4 - 2ax^2 + a^2 - a = 0$
c) $x^3 - 2ax^2 + a^2 + a = 0$
d) $x^4 - 2ax^2 + a^2 + a = 0$
e) $x^4 - 2ax^2 - 3x^2 + a^2 + 3a + 2 = 0$

29. Which of the following figures has the smallest area?

a) a square whose side is 1 unit long
b) a square whose diagonal is 1 unit long
c) a circle whose diameter is 1 unit long
d) a regular hexagon whose longest diagonal is 1 unit long
e) The smallest area occurs with two of these figures.

30. Let $a$ and $b$ be positive real numbers such that $a > b$. Which of the following statements is true?

a) $a^a b^b > a^b b^a$
b) $a^a b^b < a^b b^a$
c) $a^a b^b > a^b b^a$ if and only if $a < 2b$
d) $a^a b^b < a^b b^a$ if and only if $a < 2b$
e) none of these

31. Let $S$ be the set $\{1, 2, 3, 4, 5\}$. In how many ways can you find disjoint subsets $A$ and $B$ of $S$ so that $S = A \cup B$ and so that the sum of the numbers in $A$ is even and the sum of the numbers in $B$ is odd?

a) 8 b) 12 c) 15 d) 16 e) 32

32. Descartes' Rule of Signs says that the polynomial equation $p(x) = 0$ cannot possess more positive real roots than there are changes in sign in the coefficients of $p(x)$ and cannot possess more negative real roots than there are changes in sign in the coefficients of $p(-x)$. If we count roots according to their multiplicity, which of the following phrases best describes what Descartes' Rule of Signs tells us about the number of roots of the equation $x^6 + 4x^3 + x - 2 = 0$?

a) at most 2 real roots
b) at least 2 real roots
c) at most 4 real roots
d) at least 4 real roots
e) 6 real roots

33. How many ways can we transform the sequence $0, 0, 0, 0, 0, 0$ to the sequence $1, 1, 1, 1, 1, 1$ in six steps where each step consists of changing a single 0 to a 1?

a) 12 b) 36 c) $2^6$ d) $5!$ e) $6!$

34. Let $n = 3333 = 3 \cdot 11 \cdot 101$. For which one of the following values of $x$ is there an integer $y$ such that $xy$ has remainder 1 when divided by $n$?

a) $x = 3$ b) $x = 6$ c) $x = 11$ d) $x = 17$ e) $x = 201$

35. Suppose that $a$ and $b$ are positive integers and $a^2 - b^2 + 1$ is divisible by 4. Then

a) $a$ is even and $b$ can be either even or odd
b) $a$ is odd and $b$ can be either even or odd
c) either of $a$ and $b$ can be even if the other is odd
d) $b$ is odd and $a$ can be either even or odd
e) $a$ is even and $b$ is odd

36. For which positive integers $n$ is it the case that $n(n-1)(n-2)$ is not divisible by 12?

a) $n$ even b) $n$ odd c) $n = 3, 7, 11, 21$ only
d) $n - 3$ is divisible by 4 e) $n - 1$ is not divisible by 3

37. How many ways can the numbers $1, 2, \ldots, n$ be written so that the numbers to the left of $n$ are increasing and the numbers to the right of $n$ are decreasing?

a) $2n$ b) $2^{n-1}$ c) $n\,2^{n-1}$ d) $(n-1)!$ e) $n!$

38. Suppose you draw $n$ lines in the plane (where $n$ is even), with $k$ of them being distinct horizontal lines and the other $n - k$ being distinct vertical lines. Which value of $k$ maximizes the number of intersections between these lines?

a) $k = 1$ b) $k = 2$ c) $k = \frac{n}{2}$ d) $k = \frac{n}{2} + 1$ e) none of these

39. Consider all orderings of an ordinary deck of playing cards. What fraction of these orderings have the two of clubs preceding the three of clubs which in turn precedes the four of clubs? (Note: "preceding" does not necessarily mean immediately preceding—it just means coming somewhere before).

a) $\frac{1}{8}$ b) $\frac{1}{6}$ c) $\frac{1}{4}$ d) $\frac{1}{3}$ e) $\frac{1}{2}$

40. In a class of 30 students, the average exam score is 80. The teacher throws out the exams with top score (which was 100) and the bottom score (which was 32) and recomputes the average based on the remaining 28 exams. This new average is

a) 75.4 b) 78 c) 79 d) 81
e) There is not enough information to tell.

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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**ACKNOWLEDGMENTS**

The following individuals, corporations, and professional organizations have contributed generously to this competition:

Albion College
The Charles M. Bauervic Foundation
Ford Motor Company
The MEEMIC Foundation
The Michigan Council of Teachers of Mathematics
The Matilda R. Wilson Fund

The Michigan Association of Secondary School Principals has placed this competition on the Approved List of Michigan Contests and Activities.