

Forty-Fourth Annual
Michigan Mathematics Prize Competition

Sponsored by the
Michigan Section of the Mathematical Association of America

Part II solutions

December 6, 2000

Copyright © 2000 by the Michigan Section MAA. Permission is granted for individuals and small groups to use these questions for developing their skills in mathematical problem solving.

1. Two players picking a card from their own pile of three cards results in nine equally likely pairs of cards. The player having the higher card in five or more of these nine pairs will be more likely to pick the higher card. One such distribution of the cards is

José	2	6	10
Luciano	3	7	8
Plácido	4	5	9

Then Luciano beats José with $(3, 2)$, $(7, 2)$, $(8, 2)$, $(7, 6)$, and $(8, 6)$; Plácido beats Luciano with $(4, 3)$, $(5, 3)$, $(9, 3)$, $(9, 7)$, and $(9, 8)$; and José beats Plácido with $(6, 4)$, $(10, 4)$, $(6, 5)$, $(10, 5)$, and $(10, 9)$.

2. Suppose we had a rectangular box filled with two or more cubes each with different edge lengths. Consider the smallest cube C_1 that touches one of the faces of the box. Notice that C_1 cannot touch a pair of opposite faces of the box. If it did, the box would be too narrow to contain larger cubes. Now orient the box so a face C_1 touches is the floor. Notice that C_1 cannot be in the corner of the box touching two adjacent walls. If it were, there would be no way for it to be surrounded by the two walls and two larger cubes on the remaining two sides. Nor can C_1 touch just one of the walls. If it did, there would be no way for it to be surrounded by the wall and three larger cubes.

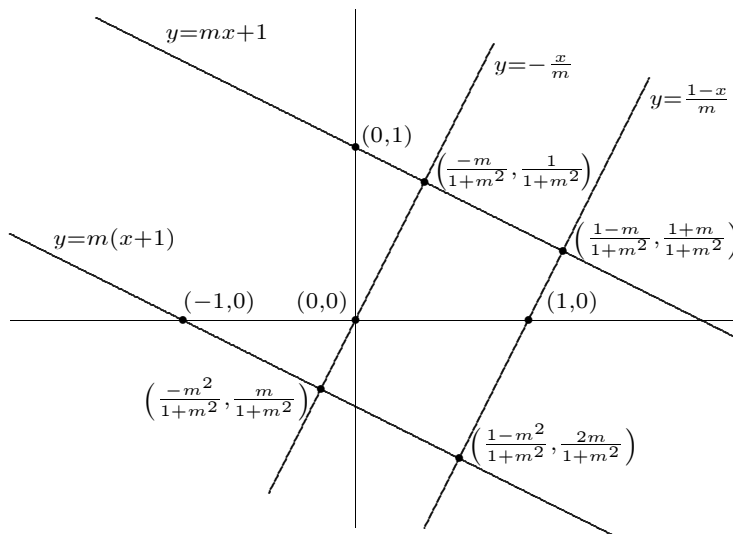
Thus the cube C_1 is surrounded by four larger cubes that form walls towering above it. Consider the smallest cube C_2 that is on top of C_1 . This cube is too small to touch a pair of opposite walls. As with C_1 , it cannot be in a corner touching two walls or along the side touching only one wall. Thus the cube C_2 is surrounded by four larger cubes that form walls towering above it.

Continuing this line of reasoning produces an infinite sequence of cubes. Thus it is impossible to fill the box with a finite number (more than two) of solid cubes each with different edge length.

3. Since vertical lines through $(0, 1)$ and $(-1, 0)$ do not result in intersection points that form a square, we can let m denote the slope of these parallel lines. Then the equation of the line through $(0, 1)$ is $y = mx + 1$, and the equation of the line through $(-1, 0)$ is $y = m(x + 1)$.

If $m = 0$, we get a square with the first two lines having equations $y = 1$ and $y = 0$.

If $m \neq 0$, then the perpendicular line through $(0, 0)$ has the equation $y = -\frac{x}{m}$, and the line through $(1, 0)$ has the equation $y = \frac{1-x}{m}$. Taking pairs of these equations, we can solve to find that the points of intersection are as indicated in the diagram.



The vertices determine a square if and only if the distances between the pairs of parallel lines are equal. Equating the squares of the distances from $(\frac{-m}{1+m^2}, \frac{1}{1+m^2})$ to $(\frac{-m^2}{1+m^2}, \frac{m}{1+m^2})$ and to $(\frac{1-m}{1+m^2}, \frac{1+m}{1+m^2})$ gives

$$\left(\frac{-m^2}{1+m^2} + \frac{m}{1+m^2}\right)^2 + \left(\frac{m}{1+m^2} - \frac{1}{1+m^2}\right)^2 = \left(\frac{1-m}{1+m^2} + \frac{m}{1+m^2}\right)^2 + \left(\frac{1+m}{1+m^2} - \frac{1}{1+m^2}\right)^2.$$

From this equation, we derive the following equivalent equations:

$$\begin{aligned} m^4 - 2m^3 + m^2 + m^2 - 2m + 1 &= 1 + m^2 \\ m^4 - 2m^3 + m^2 - 2m &= 0 \\ m^3(m - 2) + m(m - 2) &= 0 \\ (m^3 + m)(m - 2) &= 0 \\ m(m^2 + 1)(m - 2) &= 0 \end{aligned}$$

Thus the only nonzero slope is $m = 2$. This gives the pair of lines with equations $y = -2x + 1$ and $y = -2(x + 1)$.

Alternative solution using the fact that the distance from a point (r, s) to a line $ax + by + c$ is $\frac{|ar+bs+c|}{\sqrt{a^2+b^2}}$: We get a square if and only if the distance from $(0, 0)$ to $y = \frac{1-x}{m}$ is equal to the distance from $(0, 1)$ to $y = m(x + 1)$. That is, $\frac{|\frac{1}{m}|}{\sqrt{\frac{1}{m^2}+m}} = \frac{|-1+m|}{\sqrt{m^2+1}}$. This simplifies to $\frac{1}{\sqrt{1+m^2}} = \frac{|-1+m|}{\sqrt{m^2+1}}$, or $1 = |m - 1|$. Thus $m = 0$ or $m = 2$.

4. a. Suppose a_1, a_2, a_3, \dots are in arithmetic progression. For any natural number n , we have $a_{n+1} = a_n + d$, $a_{n+2} = a_n + 2d$, and $a_{n+3} = a_n + 3d$ for some constant d . Thus

$$\begin{aligned} G(n) &= (1 - \sqrt{3})a_n - (3 - \sqrt{3})a_{n+1} + (3 + \sqrt{3})a_{n+2} - (1 + \sqrt{3})a_{n+3} \\ &= (1 - \sqrt{3})a_n - (3 - \sqrt{3})(a_n + d) + (3 + \sqrt{3})(a_n + 2d) - (1 + \sqrt{3})(a_n + 3d) \\ &= (1 - \sqrt{3} - 3 + \sqrt{3} + 3 + \sqrt{3} - 1 - \sqrt{3})a_n + (-3 + \sqrt{3} + 6 + 2\sqrt{3} - 3 - 3\sqrt{3})d \\ &= 0a_n + 0d \\ &= 0. \end{aligned}$$

- b. Conversely, suppose $G(n) = 0$. Then

$$\begin{aligned} 0 = G(n) &= (1 - \sqrt{3})a_n - (3 - \sqrt{3})a_{n+1} + (3 + \sqrt{3})a_{n+2} - (1 + \sqrt{3})a_{n+3} \\ &= a_n - 3a_{n+1} + 3a_{n+2} - a_{n+3} - \sqrt{3}(a_n - a_{n+1} - a_{n+2} + a_{n+3}). \end{aligned}$$

Since $\sqrt{3}$ is not the ratio of two integers, both the integers $a_n - 3a_{n+1} + 3a_{n+2} - a_{n+3}$ and $a_n - a_{n+1} - a_{n+2} + a_{n+3}$ must equal 0. Adding these two integers yields $2a_n - 4a_{n+1} + 2a_{n+2} = 0$, or $a_n - a_{n+1} = a_{n+1} - a_{n+2}$. Since this is true for all natural numbers n , we conclude that consecutive terms of the sequence a_1, a_2, a_3, \dots differ by a constant. That is, a_1, a_2, a_3, \dots is an arithmetic progression.

5. Consider a city A that is connected by air routes to two other cities B and C . Since the Olive View Airline undoubtedly flies to the Isle of Ewe, city A might for example be the capital of this province.

We claim that the distance from B to C must be greater than the distance from A to B . Indeed, if $BC < AB$, then A is not the closest city to B , so (since A and B are connected) B must be the closest city to A . Further, since A and C are connected and since C is not the closest city to A , it follows that A must be the closest city to C . In particular, $AC < BC$. Thus we have $AC < AB$, contradicting the fact that B is the closest city to A .

Likewise, the distance from B to C must be greater than the distance from A to C . Thus, BC is the longest side in triangle ABC . Consequently the angle at A must exceed 60° . Since this is true for any two cities connected to city A , no more than five cities can be connected to A by air routes.