MMPC CODE NUMBER

FORTY-FOURTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part II

December 6, 2000

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**
2. Part II consists of problems and proofs. You will be allowed 100 minutes (1 hour and 40 minutes) for the five questions. To receive full credit for a problem, you are expected to justify your answer.
3. You are not expected to solve all the problems completely. Look over all the problems and work first on those that interest you the most. If you are unable to solve a particular problem, partial credit might be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.
4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the inside back cover (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page clearly write the **problem number**. If you use additional paper for your answer, check the appropriate box and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.
5. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.
6. The competition rules prohibit your asking questions of anyone during the examination. The use of notes, reference material, computation aids, or any other aid is likewise prohibited. Please note that **calculators are not allowed** on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. Please do not return scratch paper containing routine numerical calculations.
7. You may now open the test booklet.

#	**Score**
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$\sum$	

1. José, Luciano, and Plácido enjoy playing cards after their performances, and you are invited to deal. They use just nine cards, numbered from 2 through 10, and each player is to receive three cards. You hope to hand out the cards so that the following three conditions hold:

   A) When José and Luciano pick cards randomly from their piles, Luciano most often picks a card higher than José;

   B) When Luciano and Plácido pick cards randomly from their piles, Plácido most often picks a card higher than Luciano;

   C) When Plácido and José pick cards randomly from their piles, José most often picks a card higher than Plácido.

   Explain why it is impossible to distribute the nine cards so as to satisfy these three conditions, or give an example of one such distribution.

2. Is it possible to fill a rectangular box with a finite number of solid cubes (two or more), each with a different edge length? Justify your answer.

3. Two parallel lines pass through the points $(0, 1)$ and $(-1, 0)$. Two other lines are drawn through $(1, 0)$ and $(0, 0)$, each perpendicular to the first two. The two sets of lines intersect in four points that are the vertices of a square. Find all possible equations for the first two lines.

4. Suppose $a_1, a_2, a_3, \ldots$ is a sequence of integers that represent data to be transmitted across a communication channel. Engineers use the quantity

$$G(n) = \left(1 - \sqrt{3}\right)a_n - \left(3 - \sqrt{3}\right)a_{n+1} + \left(3 + \sqrt{3}\right)a_{n+2} - \left(1 + \sqrt{3}\right)a_{n+3}$$

   to detect noise in the signal.

   a. Show that if the numbers $a_1, a_2, a_3, \ldots$ are in arithmetic progression, then $G(n) = 0$ for all $n = 1, 2, 3, \ldots$.

   b. Show that if $G(n) = 0$ for all $n = 1, 2, 3, \ldots$, then $a_1, a_2, a_3, \ldots$ is an arithmetic progression.

5. The Olive View Airline in the remote country of Kuklafrania has decided to use the following rule to establish its air routes: If A and B are two distinct cities, then there is to be an air route connecting A with B either if there is no city closer to A than B or if there is no city closer to B than A. No further routes will be permitted. Distances between Kuklafranian cities are never equal. Prove that no city will be connected by air routes to more than five other cities.

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

DIRECTOR

Robert Messer
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