FORTY-FOURTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part I

October 11, 2000

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. Check to ensure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please quit working immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.

5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "e) none of these". If you believe none of the first four alternatives to be correct, mark e) in such cases.

6. Any scientific or graphing calculator is permitted on Part I. Unacceptable machines include portable computers and pocket organizers. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator.

7. No one is permitted to explain to you the meaning of any question. Do not request anyone to break the rules of the competition. If you have questions concerning the instructions, ask them now.

8. You may now open the test booklet and begin.

1. The Nasdaq Composite Index fell 20% from May 1 to May 31, and then gained 15% from June 1 to June 30. What happened to the Nasdaq Composite Index from May 1 to June 30?

a) fell 8% b) fell 5% c) neither gained nor lost d) gained 2% e) gained 5%

2. The average of sixty numbers is 24. Two of the numbers are discarded, and the average of the remaining 58 numbers is 25. Then the sum of the two discarded numbers is

a) -20 b) -10 c) 0 d) 10 e) none of these

3. Dan paid \$9.95 for one round pizza with 38 cm diameter. Bill paid \$8.95 for two round pizzas with 26 cm diameters. Phil paid \$7.95 for three round pizzas with 20 cm diameters. Ed paid \$6.95 for four round pizzas with 16 cm diameters. Based on price per square centimeter, who got the best deal?

a) Dan b) Bill c) Phil d) Ed e) two of them are tied

4. Two integers are chosen randomly between 0 and 80 inclusive. The sum of these integers is divided by 9. What is the probability that the remainder is 3?

a) $\frac{1}{9}$ b) $\frac{2}{9}$ c) $\frac{1}{3}$ d) $\frac{1}{2}$ e) $\frac{2}{3}$

5. If $f(x) = \dfrac{x-1}{x+1}$, what is the value of $f(f(f(2)))$?

a) -3 b) $-\frac{1}{2}$ c) $\frac{1}{3}$ d) 3 e) none of these

6. At a campground near Lake Michigan, the temperature outside a cabin is $0.8t + 18$ degrees Celsius, and the temperature inside is $1.5t + 15$ degrees Celsius, where t is the number of hours after 6 am. To the nearest minute, at what time will the temperatures inside and outside be the same?

a) 6:04 am b) 10:17 am c) 10:29 am d) it is always warmer outside
e) none of these

7. Let q equal the sum of all the positive divisors of 957 and r equal the sum of all the positive divisors of 958. Which of the following statements is true?

a) $q > r$ b) q is prime c) 957 has 7 divisors d) r is prime e) none of these

8. Three distinct numbers are chosen at random from the set

$$\{4, 8, 13, 16, 24, 26, 30, 33, 38, 44, 46, 47, 51, 64, 83\}.$$

What is the probability that none of the chosen numbers is odd?

a) $\frac{97}{531}$ b) $\frac{33}{91}$ c) $\frac{45}{91}$ d) $\frac{24}{91}$ e) none of these

9. When a rectangular ditch 2 m wide and 1 m deep was dug around (and adjacent to) the foundation of a square building, the volume of soil removed was $200\,\mathrm{m}^3$. Find the length of the side of the building.

a) 23 m b) $20\sqrt{2}$ m c) 46 m d) 49 m e) 50 m

10. A triangle with sides of lengths 6, 8, and 10 is inscribed in a circle. What is the radius of the circle?

a) $2\sqrt{3}$ b) $3\sqrt{2}$ c) $2\sqrt{5}$ d) $2\sqrt{6}$ e) none of these

11. The graph of the equation $\sqrt{(x-4)^2+y^2}+\sqrt{x^2+(y+3)^2}=7$ is

a) an ellipse b) a parabola c) a pair of lines d) a hyperbola e) none of these

12. Suppose the line L_1 has equation $10x - 12y = -17$. The line L_2 intersects L_1 at $\left(\frac{1}{2}, \frac{11}{6}\right)$ and is perpendicular to L_1. Determine the x-coordinate of the point on L_2 whose y-coordinate is $\frac{1}{30}$.

a) 1 b) $\frac{179}{150}$ c) 2 d) $\frac{359}{150}$ e) none of these

13. By choosing different real values for a in the system of equations $\begin{cases} x^2 - y^2 = 0 \\ (x-a)^2 + y^2 = 1 \end{cases}$ the number of distinct real-valued solutions can be

a) 0, 1, 2, 3, 4, or 5 b) 0, 1, 2, or 4 c) 0, 2, 3, or 4
d) 0, 2, or 4 e) 2 or 4

14. A certain polyhedron has 20 faces, all of them triangles. Each vertex of the polyhedron is the common vertex of 5 of the triangular faces, and each edge is shared by 2 of the faces. How many vertices does the polyhedron have?

a) 10 b) 12 c) 15 d) 20 e) 24

15. A motor boat moves in a direction 40 degrees east of due north at 20 km/hr for 3 hours. How far north does it travel (to the nearest kilometer)?

a) 39 km b) 44 km c) 45 km d) 46 km e) 50 km

16. Let $a_0, a_1, a_2, \ldots$ be a sequence of numbers. Suppose that the first number is $a_0 = 7$ and that $a_n - a_{n-1} = 3$ for all $n \geq 1$. Which is the formula for the numbers a_n?

a) $a_n = 3n + 7$ b) $a_n = 7n + 3$ c) $a_n = n^2 + 3n + 7$ d) $a_n = n^2 + 7n + 3$

e) It is impossible to tell from the information given.

17. In how many ways can you place eight identical rooks on a standard eight-by-eight chessboard so that no two of the rooks attack each other (i.e., so that no two rooks are in the same row or in the same column)?

a) $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36$ b) 8^2 c) 2^8 d) $8!$ e) 8^8

18. For any given constant k, what is the smallest value that $f(x) = 4x^2 + 16x + k$ achieves?

a) $k - 16$ b) -2 c) k d) $k + 48$ e) there is no smallest value

19. The expression $(\cos 2\alpha)(\sec^2 \alpha)$ is equivalent to

a) 1 b) $\sec \alpha$ c) $1 + \tan^2 \alpha$ d) $1 - \tan^2 \alpha$ e) none of these

20. If the repeating decimal $1.38383838\ldots$ is written as a quotient of two positive integers reduced to lowest terms, then the sum of the numerator and denominator is

a) 137 b) 236 c) 238 d) 1384 e) none of these

21. What is the remainder when $x^{10} - 1$ is divided by $x^2 - a$?

a) $\frac{1}{a}$ b) $a - 1$ c) $a^5 - 1$ d) $a^{10} - 1$ e) none of these

22. A trapezoid has three sides of length 2 and one side of length 1. What is its area?

a) $\sqrt{7}$ b) $\frac{3}{4}\sqrt{15}$ c) 3 d) $\frac{7}{2}$ e) none of these

23. If $\text{Arctan}\, x + \text{Arctan}\, a = 45°$, then x is

a) $1 + a$ b) $1 - a$ c) $1 - a^2$ d) $\dfrac{1-a}{1+a}$ e) none of these

24. If $\log_{10}(bc) < -1$, then which of the following must be true?

a) $bc > 1$ b) $bc < \frac{1}{100}$ c) $bc < -\frac{1}{10}$ d) $|b| + |c| > 0$ e) none of these

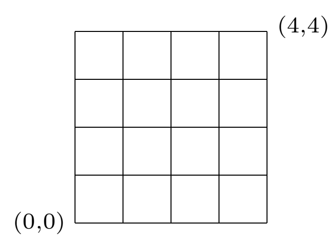
25. A regular polygon has 44 diagonals. How many sides does it have?

a) 8 b) 9 c) 10 d) 11 e) 12

26. A crew rows four miles downstream and back the same distance in one hour. If the stream flows at 3 miles per hour, the crew's rate of rowing in still water would be (in miles per hour)

a) 8 b) 9 c) 10 d) 11 e) 12

27. Paths of length 8 are traced along the horizontal and vertical lines of the rectangular grid shown in the diagram, beginning at $(0, 0)$ and ending at $(4, 4)$. If all such paths are equally likely, what is the probability that such a path will pass through $(2, 2)$?



a) $\frac{15}{32}$ b) $\frac{1}{2}$ c) $\frac{18}{35}$ d) $\frac{17}{32}$ e) $\frac{19}{35}$

28. Let k be a positive integer. Which of the following kinds of integers can be a perfect square?

a) $10^k - 3$ b) $10^k - 2$ c) $10^k + 2$ d) $10^k + 3$ e) none of these

29. One of the corners of a square with side lengths 12 cm is the center of a circle of radius q cm. For what value of q (rounded to three decimal places) will the part of the square inside the circle have the same area as the part outside the circle?

a) 4.787 b) 8.485 c) 9.575 d) 12.000 e) none of these

30. The number $5! = 120 = 2^3 \cdot 3 \cdot 5$ has 3 different prime factors. How many different prime factors does 21! have?

a) 7 b) 8 c) 10 d) 18 e) 19

31. Consider the following three hands of five cards dealt from an ordinary deck of 52 cards:

A) all four queens and a two,
B) the ace, king, queen, jack, and ten all in the same suit, and
C) the ten of hearts, the ten of clubs, and three kings.

a) A is more likely than B or C.
b) B is more likely than A or C.
c) C is more likely than A or B.
d) Two of the three hands are equally likely and more likely than the third.
e) All three hands are equally likely.

32. If the logarithm of 27 to the base b is 1.5, then b is

a) 2 b) e c) 3 d) 9 e) 10

33. Four different integers w, x, y, z satisfy the equation

$$(7-w)(7-x)(7-y)(7-z) = 4.$$

What is the value of $w+x+y+z$?

a) 10 b) 21 c) 24 d) 26 e) 28

34. Determine the last two digits of 444^{88}.

a) 04 b) 16 c) 44 d) 64 e) none of these

35. In the equation $21x - 1101 = 2000$, the coefficients are written in base 3. What is the solution of this equation, written in base 3?

a) 110 b) 111 c) 112 d) 121 e) 201

36. A circle is inscribed in an equilateral triangle. Which of the following is nearest the percent of the triangle's area occupied by the circle?

a) 50% b) 55% c) 60% d) 65% e) 70%

37. A molecule of DNA consists of a sequence of bases denoted by the letters A, C, G, and T. In a living cell consecutive triples of bases represent the various kinds of molecules called amino acids (for example, the triple GCA represents the amino acid alanine). There are twenty kinds of amino acids. There are

a) five times as many amino acids as there are triples to represent them.
b) approximately twice as many amino acids as there are triples to represent them.
c) exactly the same number of amino acids as there are triples to represent them.
d) approximately three times as many triples as there are amino acids.
e) more than ten times as many triples as there are amino acids.

38. A geometric figure has four vertices, A, B, C, and D, and the following distances between the vertices: $|AB| = 1$, $|AC| = 1$, $|AD| = \frac{11}{20}$, $|BC| = 1$, $|BD| = \frac{11}{20}$, and $|CD| = \frac{11}{20}$. This figure

a) is a triangle b) is a square c) is a quadrilateral other than a square
d) is a tetrahedron e) does not exist

39. A polynomial with roots -4, 3, and 24 is of the form $x^3 + ax^2 - bx + c$. Which of the following statements is true?

a) a is the sum of the roots b) $b = -36$ c) c is the product of the roots
d) a is the product of the roots e) none of these

40. How many miles does a train travel in 12 hours if it averages 40 mph when moving and makes n stops of m minutes each?

a) $480 - 40mn$ b) $1440 - 2mn$ c) $\dfrac{480 - 2mn}{3}$ d) $\dfrac{1440 - 2mn}{3}$

e) none of these

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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Robert Messer
Albion College

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