MMPC CODE NUMBER

FORTY-THIRD ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part II

December 8, 1999

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**

2. Part II consists of problems and proofs. You will be allowed 100 minutes (1 hour and 40 minutes) for the five questions. To receive full credit for a problem, you are expected to justify your answer.

3. You are not expected to solve all the problems completely. Look over all the problems and work first on those that interest you the most. If you are unable to solve a particular problem, partial credit might be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.

4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the inside back cover (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page clearly write the **problem number**. If you use additional paper for your answer, check the appropriate box and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.

5. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.

6. The competition rules prohibit your asking questions of anyone during the examination. The use of notes, reference material, computation aids, or any other aid is likewise prohibited. Please note that calculators are **not** allowed on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. Please do not return scratch paper containing routine numerical calculations.

7. You may now open the test booklet.

#	Score
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1. (10 pts) The final Big 10 standings for the 1996 Women's Softball season were

1. Michigan	6. Purdue
2. Minnesota	7. Northwestern
3. Iowa	8. Ohio State
4. Indiana	9. Penn State
5. Michigan State	10. Wisconsin

(Illinois does not participate in Women's Softball.)

When you compare the 1996 final standings (above) to the final standings for the 1999 season, you find that the following pairs of teams changed order relative to each other from 1996 to 1999 (there are no ties, and no other pairs changed places):

(Iowa, Michigan State)	(Indiana, Penn State)	(Purdue, Wisconsin)
(Iowa, Penn State)	(Indiana, Wisconsin)	(Northwestern, Penn State)
(Indiana, Michigan State)	(Michigan State, Penn State)	(Northwestern, Wisconsin)
(Indiana, Purdue)	(Purdue, Northwestern)	(Ohio State, Penn State)
(Indiana, Northwestern)	(Purdue, Penn State)	(Ohio State, Penn State)
(Indiana, Ohio State)		

Determine as much as you can about the final Big 10 standings for the 1999 Women's Softball season. If you cannot determine the standings, explain why you do not have enough information. You must justify your answer.

2. a) (5 pts) Take as a given that any expression of the form

$$A\sin t + B\cos t \qquad (A > 0)$$

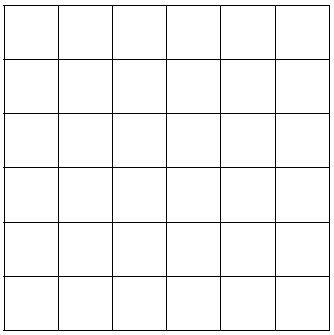
can be put in the form $C\sin(t + D)$, where $C > 0$ and ¡¼/2 < D < ¼/2. Determine C and D in terms of A and B.

b) (4 pts) For the values of C and D found in part a), prove that

$$A\sin t + B\cos t = C\sin(t + D).$$

c) (1 pt) Find the maximum value of $3\sin t + 2\cos t$.

3. (10 pts) A 6-by-6 checkerboard is completely filled with 18 dominoes (blocks of size 1-by-2). Prove that some horizontal or vertical line cuts the board in two parts but does not cut any of the dominoes.



4. a) (4 pts) The midpoints of the sides of a regular hexagon are the vertices of a new hexagon. What is the ratio of the area of the new hexagon to the area of the original hexagon? Justify your answer and simplify as much as possible.

b) (6 pts) The midpoints of the sides of a regular n-gon ($n > 2$) are the vertices of a new n-gon. What is the ratio of the area of the new n-gon to that of the old? Justify your answer and simplify as much as possible.

5. You run a boarding house that has 90 rooms. You have 100 guests registered, but on any given night only 90 of these guests actually stay in the boarding house. Each evening a different random set of 90 guests will show up. You don't know which 90 it will be, but they all arrive for dinner before you have to assign rooms for the night. You want to give out keys to your guests so that for any set of 90 guests, you can assign each to a private room without any switching of keys.

a) (3 pts) You could give every guest a key to every room. But this requires 9000 keys. Find a way to hand out fewer than 9000 keys so that each guest will have a key to a private room.

b) (7 pts) What is the smallest number of keys necessary so that each guest will have a key to a private room? Describe how you would distribute these keys and assign the rooms. Prove that this number of keys is as small as possible.

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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