Forty-Second Annual

# Michigan Mathematics Prize Competition

Sponsored by the

Michigan Section of the Mathematical Association of America

## Part I

October 14, 1998



1. Let $f(x) = 1 + x$ and let $g(x) = 1/x$. How many different real number satisfy the equation $x = g(f(x))$?

   a) 0 b) 1 c) 2 d) 3 e) infinitely many

2. You attend a party at which every individual has exactly four friends of the opposite sex. Fifteen men attend the party. How many women attend?

   a) 4 b) 15 c) 19 d) 60 e) can't tell from the information given

3. The sum of all of the coefficients in the expansion of the binomial $(2x + y)^5$ is

   a) 1024 b) 243 c) 64 d) 32 e) some other number

4. For what value(s) of $k$ will the equation $2x^2 - kx + x + 8$ have equal real roots?

   a) 9 and $-7$ b) only 7 c) 9 and 7 d) $-9$ and $-7$ e) only 9

5. Thirty-five workers can do a job in fifteen 8-hour days. The number of workers required to do the same job in twenty-four 7-hour days is

   a) 15 b) 21 c) 25 d) 35 e) none of these

6. What digit is in the 1's place of $3^{14829}$?

   a) 1 b) 3 c) 7 d) 8 e) 9

7. Let $u(x) = e^{-x^2}$. On which interval is $u(x)$ increasing?

   a) $x \in [\,0, 1\,]$ b) all $x$ greater that 0 c) all $x$ d) all $x$ less than 0
   e) $x \in [\,-1, 1\,]$

8. A line through the midpoints of two sides of a triangle divides the area into two parts. The ratio of the larger areas to the smaller is

   a) 2 b) $2\sqrt{2}$ c) 3 d) $2\sqrt{3}$ e) depends on the triangle

9. Suppose 21 people take an exam and their average score is 75. Suppose that one person got a 55. What is the highest score that you can be sure that someone got on the exam?

   a) 54 b) 55 c) 75 d) 76 e) 95

10. Which is the largest?

    a) 11100 base 2 b) 1001 base 3 c) 103 base 5 d) 26 base 11
    e) they are all equal

11. A point $P$ is outside a circle and is 13 cm from its center. A line drawn from $P$ cuts the circle at $Q$ and $R$ so that $\overline{PQ}$ is 9 cm and $\overline{QR}$ is 7 cm long. What is the radius of the circle (in cm)?

    a) 3 b) 4 c) 5 d) 6 e) none of these

12. Let $N$ be the product of the first 200 positive integers. The largest integer $k$ such that $3^k$ is a factor of $N$ is:

    a) 70 b) 88 c) 95 d) 97 e) some other number

13. How many distinct orderings are there of the four letters $M$, $M$, $P$, $C$?

    a) 6 b) 8 c) 12 d) 24 e) none of these

14. You arrived at a bus stop at 9 am. You know that buses run at 10-minute intervals, but you know nothing about when the next one is scheduled to arrive. How long is your average wait for the next bus?

a) 1 minute b) 5 minutes c) 6 minutes d) 10 minutes
e) none of these

15. For what values of $x$ are $2^x - x + 3$ and $7 - x$ equal?

a) just $x = 2$ b) $x = \pm 2$ c) all values of $x$ d) no values of $x$
e) none of these

16. For which values of $n$ is $x + 1$ a factor of $x^n - 1$?

a) just $n = 2$ b) just $n = 3$ c) no values of $n$ d) all positive values of $n$
e) all positive even values of $n$

17. The result of raising 3 to the power $\dfrac{\log_4 8}{\log_4 3}$ is

a) $\frac{8}{3}$ b) 3 c) 4 d) 8 e) none of these

18. A long strip of paper 0.01 cm thick and 10 cm wide is wound tightly about a wooden cylinder 5 cm in diameter. The resulting reel of paper is 10 cm in diameter. Which of the following numbers is closest to the length of the strip of paper (in cm)?

a) 4700 b) 5100 c) 5500 d) 5900 e) 6300

19. If $0 < b < a$ and if $ax + \dfrac{b}{y} = bx + \dfrac{a}{y} = 1$, then $xy$ equals

a) $\dfrac{b}{a}$ b) $\dfrac{a}{b}$ c) 1 d) $-1$ e) $\dfrac{b}{a} + \dfrac{a}{b}$

20. A student walks 5 blocks east and 3 blocks north from home to school each morning. In how many different ways can the student make the trip, assuming that the student walks the minimum distance possible?

a) 5 b) 15 c) $3^5$ d) $5^3$ e) 56

21. What is the smallest integer that is greater than 1, is not a prime, and is not divisible by any one of 2, 3, 5, 7, or 11?

a) 13 b) 221 c) $2311 = (2 \cdot 3 \cdot 5 \cdot 7 \cdot 11) + 1$ d) 169 e) none of these

22. How many real numbers $x$ satisfy the equation $x^{2\log_2 x} = 8$?

a) none b) 1 c) 2 d) 4 e) more than 4

23. Find the value of $\tan x$ if $\sin x + \cos x = \frac{1}{5}$ and $\frac{\pi}{2} < x < \pi$.

a) $-\frac{4}{3}$ b) $\frac{-4+\sqrt{7}}{4}$ c) $\frac{-3+\sqrt{7}}{4}$ d) $\frac{-5+\sqrt{7}}{4}$ e) $\frac{-5+\sqrt{7}}{3}$

24. Consider the following matrix of numbers and letters:

$$\begin{matrix} 1 & D & 11 \\ 8 & D & B \\ A & D & C \end{matrix}$$

Suppose that $A, B, C, D$ are assigned numerical values so that every row and column in the resulting matrix has the same sum. What value is assigned to $A$?

a) 3 b) 6 c) 8 d) 9 e) none of these

25. Let $R$ be a rectangle with sides of lengths $x$ and $y$ $(x, y > 0)$ and diagonal of length $z$. Which of the following is largest?

a) $z^2$ b) $(x+y)^2$ c) $2 \cdot \text{area}(R)$ d) $x^2 + xy$
e) sometimes a) and sometimes b), depending on $R$

26. In triangle $ABC$, angle $A$ measures $42^\circ$, while angle $B$ measures $50^\circ$. Which of the following statements is not true?

a) $\overline{AB} > \overline{BC}$ b) $\overline{AC} > \overline{BC}$ c) $\overline{AC} > \overline{AB}$ d) angle $C >$ angleB
e) angle $C >$ angleA

27. Consider the following options for receiving money:
Option 1: Someone gives you 1,000,000 dollars.
Option 2: Someone pays you 100 dollars every day for 10 years.
Option 3: Someone pays you 50 dollars every day for 25 years.
Option 4: Someone pays you 1 dollar today, and then every day for the next 30 days pays you twice as much as you received the day before.

Under which of these options would you receive the most money?

a) Option 1 b) Option 2 c) Option 3 d) Option 4
e) you receive exactly the same amount with each of these options

28. You roll three fair dice. What is the probability that the three numbers you roll add up to twelve?

a) $\frac{1}{36}$ b) $\frac{6}{36}$ c) $\frac{12}{216}$ d) $\frac{24}{216}$ e) $\frac{25}{216}$

29. What do you get when you add up the $n$ smallest positive odd numbers?

a) $2n-1$ b) $2^{n(n-1)/2}$ c) $n!$ d) $n^2$ e) none of these

30. How many real numbers $x$ satisfy the equation $x^{(x+1)^2} = x^{16}$?

a) 0 b) 1 c) 2 d) 4 e) more than 4

31. A ball of radius 1 sits in a right circular cone (whose vertex points down) having vertex angle $40^\circ$. The distance from the cone's vertex to the top of the ball is approximately

a) 2.56 b) 2.75 c) 2.92 d) 3.75 e) 3.92

32. Sam lives with Brad. Sally lives east of Mary and west of Brad. Ed lives east of Sam and west of Mary. If these five people live in California, Colorado, Illinois, Ohio, and Massachusetts, where does Sam live?

a) California b) Colorado c) Illinois d) Massachusetts e) Ohio

33. A parabolic dish antenna is 10 cm deep and 100 cm in diameter. How far from its vertex is its focus?

a) 62.5 cm b) 50 cm c) 31.25 cm d) 6.25 cm e) 5 cm

34. Find the size of the acute angle between the hands of a clock that shows the time to be 1:20.

a) $75^\circ$ b) $80^\circ$ c) $90^\circ$ d) $120^\circ$ e) none of these

35. If $S_n = 1 - 2 + 3 - 4 + \cdots + (-1)^{n+1}n$, then $S_{1998} =$?

a) $-1998$ b) $-999$ c) $0$ d) $999$ e) $1998$

36. Two different numbers are selected at random from the set $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$. What is the probability that their sum is 24?

a) $\frac{1}{30}$ b) $\frac{2}{45}$ c) $\frac{1}{15}$ d) $\frac{1}{10}$ e) $\frac{2}{15}$

37. For which values of $n > 1$ is $\log_n(10) > \log_{10} n$?

a) $n > 2$ b) $n = 10$ c) $n > 10$ d) $n < 10$ e) none of these

38. Let $T$ be a trapezoid (a quadrilateral having at least one pair of parallel sides) with sides of length $a$, $x$, $b$, $y$ as you view the sides in clockwise order. Engineers in ancient Egypt used the quantity $E = \dfrac{a+b}{2} \cdot \dfrac{x+y}{2}$ to estimate the area of $T$. Which of the following statements is correct?

a) $E$ is always an underestimate for the actual area.

b) $E$ is always an overestimate for the actual area.

c) $E$ is always greater than or equal to the actual area and equal to the area for some trapezoids.

d) $E$ is always equal to the area of $T$.

e) $E$ is an underestimate for some trapezoids and an overestimate for others.

39. Let $i$ denote $\sqrt{-1}$. Evaluate $i + i^2 + i^3 + i^4 + \cdots + i^{150}$.

a) $1$ b) $i$ c) $i + 1$ d) $i - 1$ e) none of these

40. Simplify $\sqrt[3]{\sqrt{10} + \sqrt{2}} \cdot \sqrt[3]{\sqrt{10} - \sqrt{2}}$ as much as possible.

a) $1$ b) $8$ c) $\sqrt[3]{2}$ d) $\sqrt[6]{\sqrt{10} \pm \sqrt{2}}$ e) none of these

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## Part I

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## Answers

| | | | |
|---|---|---|---|
| 1. c | 11. c | 21. d | 31. e |
| 2. b | 12. d | 22. c | 32. a |
| 3. b | 13. c | 23. a | 33. a |
| 4. a | 14. b | 24. d | 34. b |
| 5. c | 15. a | 25. b | 35. b |
| 6. b | 16. e | 26. c | 36. c |
| 7. d | 17. d | 27. d | 37. d |
| 8. c | 18. d | 28. e | 38. c |
| 9. d | 19. c | 29. d | 39. d |
| 10. e | 20. e | 30. e | 40. e |