Arrangements of Stars on the American Flag

Johann Thiel - City Tech
Dimitris Koukoulopoulos - CRM

April 28, 2024
The Union Jack

- The Jack of the United States, or **Union Jack**, is the blue portion of the American flag containing one star for each state.

![Current Union Jack](image)

- From 1777 to 2002, the Union Jack was the official maritime flag representing the United States.
Puerto Rico

- In Chris Wilson’s Slate article, *13 Stripes and 51 Stars*, he mentions the possibility that Puerto Rico may vote to become the 51\textsuperscript{st} state.

- **Problem:** How do we add an additional star to the Union Jack so that it looks “nice?”

- How was this problem resolved in 1959 and 1960?
- Robert G. Heft!
Long - 50 Stars

Short - 42 Stars

Equal - 48 Stars

Alternate - 45 Stars

Wyoming - 32 Stars

Oregon - 33 Stars
1 to 100 Stars

- Heft designed arrangements for a flag with 51 to 60 stars.
- Skip Garibaldi created a program that finds arrangements for 1 to 100 stars using the arrangements from the previous slide.
- The \textit{N} States of America (no longer working)
- What about 29, 69, and 87?
The Problem with 29

Let $a$ and $b$ represent the number of rows and columns of stars on a 29-star flag.

- For the equal arrangement, we need $29 = ab$ with $1 \leq b/a \leq 2$.
- For the Oregon arrangement, we need $31 = 29 + 2 = ab$ with $1 \leq b/a \leq 2$.
- For the Wyoming arrangement, we need $27 = 29 - 2 = ab$ with $a$ and $b$ close to each other.
- For the remaining arrangements (long, short), we need 59 or 57 to factor as a product $ab$ with $a$ and $b$ close to each other.
Characterization of Arrangements

A nice arrangement of \( n \) stars on the Union Jack exists if at least one of the following holds:

(i) For the long pattern, \( 2n - 1 = (2a + 1)(2b + 1) \) with \( 1 \leq (b + 1)/(2a + 1) \leq 2 \).

(ii) For the short pattern, \( 2n + 1 = (2a - 1)(2b + 1) \) with \( 1 \leq (b + 1)/(2a + 1) \leq 2 \).

(iii) For the alternate pattern, \( n = a(2b - 1) \) with \( 1 \leq b/(2a) \leq 2 \).

(iv) For the Wyoming pattern, \( n - 2 = ab \) with \( 1 \leq (b + 1)/a \leq 2 \).

(v) For the equal pattern, \( n = ab \) with \( 1 \leq b/a \leq 2 \).

(vi) For the Oregon pattern, \( n + 2 = ab \) with \( 1 \leq b/a \leq 2 \).
Notation

We write \( f(N) = O(g(N)) \) if

\[
|f(N)| \leq c|g(N)|
\]

for some constant \( c > 0 \) as \( N \to \infty \).

For a set of non-negative integers \( A \), we call

\[
\lim_{N \to \infty} \frac{\#\{n \leq N : n \in A\}}{N}
\]

the **asymptotic density** of \( A \).
Let $\Omega(n) = \sum_{\substack{p^a|n \\ a \geq 1}} 1$.

$\Omega(20) = \Omega(4) + \Omega(5) = 2 + 1 = 3$

One can show that $\frac{1}{N} \sum_{n \leq N} \Omega(n) = \log \log N + O(1)$.

For $n \leq 10^{100}$, $\Omega(n) \approx 6$. 
Sketch of the proof for $\Omega(n)$

**Theorem (Mertens)**

*We have that*

$$\sum_{p \leq N} \frac{1}{p} = \log \log N + O(1).$$
Sketch of the proof for $\Omega(n)$

From Mertens' result, it follows that

$$\frac{1}{N} \sum_{n \leq N} \Omega(n) = \frac{1}{N} \sum_{n \leq N} \sum_{p^a | n, a \geq 1} 1$$

$$= \frac{1}{N} \sum_{p^a \leq N, n \leq N} \sum_{a \geq 1} 1$$

$$= \frac{1}{N} \sum_{p^a \leq N} \left( \frac{N}{p^a} + O(1) \right)$$

$$= \sum_{p \leq N} \frac{1}{p} + O \left( \sum_{p \text{ prime}} \frac{1}{p^a} \right) \quad = \log \log N + O(1).$$
Theorem (Hardy, Ramanujan)

For any $\epsilon > 0$, 

$$ \lim_{N \to \infty} \frac{1}{N} \# \left\{ n \leq N : |\Omega(n) - \log \log N| \leq \epsilon \log \log N \right\} = 1. $$
## The Multiplication Table Problem

<table>
<thead>
<tr>
<th>×</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>6</td>
<td>9</td>
<td>12</td>
<td>15</td>
<td>18</td>
<td>21</td>
<td>24</td>
<td>27</td>
<td>30</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>20</td>
<td>24</td>
<td>28</td>
<td>32</td>
<td>36</td>
<td>40</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>25</td>
<td>30</td>
<td>35</td>
<td>40</td>
<td>45</td>
<td>50</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>12</td>
<td>18</td>
<td>24</td>
<td>30</td>
<td>36</td>
<td>42</td>
<td>48</td>
<td>54</td>
<td>60</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>14</td>
<td>21</td>
<td>28</td>
<td>35</td>
<td>42</td>
<td>49</td>
<td>56</td>
<td>63</td>
<td>70</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>16</td>
<td>24</td>
<td>32</td>
<td>40</td>
<td>48</td>
<td>56</td>
<td>64</td>
<td>72</td>
<td>80</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>18</td>
<td>27</td>
<td>36</td>
<td>45</td>
<td>54</td>
<td>63</td>
<td>72</td>
<td>81</td>
<td>90</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>50</td>
<td>60</td>
<td>70</td>
<td>80</td>
<td>90</td>
<td>100</td>
</tr>
</tbody>
</table>
The Multiplication Table Problem

<table>
<thead>
<tr>
<th>×</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>20</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>15</td>
<td>21</td>
<td>24</td>
<td>27</td>
<td>30</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>28</td>
<td>32</td>
<td>36</td>
<td>40</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>25</td>
<td>35</td>
<td>45</td>
<td>50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>42</td>
<td>48</td>
<td>54</td>
<td>60</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>49</td>
<td>56</td>
<td>63</td>
<td>70</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>64</td>
<td>72</td>
<td>80</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>81</td>
<td>90</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>100</td>
</tr>
</tbody>
</table>
Heuristic argument

- Let \( A(N) = \#\{n \leq N : n = n_1 n_2, n_i \leq \sqrt{N}\} \)
- Suppose \( n \) is in the multiplication table where the axis ranges from 1 to \( \sqrt{N} \).
- \( n = n_1 n_2 \Rightarrow \Omega(n) = \Omega(n_1) + \Omega(n_2) \approx 2 \log \log \sqrt{N} \)
- This implies that \( \Omega(n) \approx 2 \log \log N \), which is not very common.
  
  Note: \( \log \log \sqrt{N} = \log \log N + \log \log 1/2 \approx \log \log N \).
- We expect \( A(N) \) to be small.
A more complete argument

Suppose that $n$ is an integer counted by $A(N)$. Then $n = n_1n_2$ with $n_1, n_2 \leq \sqrt{N}$ and either

Case 1: $\Omega(n_1) < \frac{2}{3} \log \log N$ or

Case 2: $\Omega(n_2) \geq \Omega(n_1) \geq \frac{2}{3} \log \log N \Rightarrow \Omega(n) \geq \frac{4}{3} \log \log N$.

The number of integers $n \leq N$ counted by Case 1 is at most

\[
\# \left\{ n_1 \leq \sqrt{N} : \Omega(n_1) \leq \frac{2}{3} \log \log N \right\} \cdot \# \left\{ n_2 \leq \sqrt{N} \right\}.
\]

The number of integers $n \leq N$ counted by Case 2 is at most

\[
\# \left\{ n \leq N : \Omega(n) \geq \frac{4}{3} \log \log N \right\}.
\]
A more complete argument

So

\[
\frac{A(N)}{N} \leq \frac{1}{\sqrt{N}} \# \left\{ n_1 \leq \sqrt{N} : \Omega(n_1) \leq \frac{2}{3} \log \log N \right\} \\
+ \frac{1}{N} \# \left\{ n \leq N : \Omega(n) \geq \frac{4}{3} \log \log N \right\} \\
\rightarrow 0
\]

as \( N \rightarrow \infty \).
Results on \( A(N) \)

Erdős (1960)

\[
A(N) = O_\epsilon \left( \frac{N}{(\log N)^{\delta - \epsilon}} \right), \quad \delta = 1 - \frac{1 + \log \log 2}{\log 2} = 0.086071 \ldots
\]

Hall–Tenenbaum (1988)

\[
A(N) = O \left( \frac{N}{(\log N)^{\delta} \sqrt{\log \log N}} \right)
\]

Ford (2008)

\[
A(N) \asymp \frac{N}{(\log N)^{\delta} (\log \log N)^{3/2}}
\]
Equal Patterns

Let $n \leq N$ be such that it admits an equal star arrangement.

So $n = ab$ with $1 \leq \frac{b}{a} \leq 2$.

$a \leq b \leq 2a \Rightarrow \sqrt{n} \leq b \leq 2\sqrt{n}$.

In particular, $a, b \leq 2\sqrt{N}$, so $n$ is counted by $A(4N)$.

This implies that such $n$ have asymptotic density 0.

Note: We can argue in a similar way for the remaining arrangements.
Theorem (Koukoulopoulos, T)

The set of non-negative integers allowing for a nice arrangement of $n$ stars on the U.S. flag has asymptotic density zero.
Thank you!