Arrangements of Stars on the American Flag

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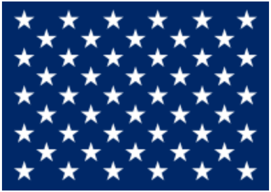
The Union Jack

- The Jack of the United States, or **Union Jack**, is the blue portion of the American flag containing one star for each state.



Current Union Jack

- From 1777 to 2002, the Union Jack was the official maritime flag representing the United States.

Puerto Rico

- In Chris Wilson's Slate article, *13 Stripes and 51 Stars*, he mentions the possibility that Puerto Rico may vote to become the 51st state.
- **Problem:** How do we add an additional star to the Union Jack so that is looks "nice?"

- How was this problem resolved in 1959 and 1960?
- Robert G. Heft!

Long - 50 Stars

Short - 42 Stars

Equal - 48 Stars

Alternate - 45 Stars

Wyoming - 32 Stars

Oregon - 33 Stars

1 to 100 Stars

- Heft designed arrangements for a flag with 51 to 60 stars.
- Skip Garibaldi created a program that finds arrangements for 1 to 100 stars using the arrangements from the previous slide.
- The N States of America (no longer working)
- What about 29, 69, and 87?

The Problem with 29

Let a and b represent the number of rows and columns of stars on a 29-star flag.

- For the equal arrangement, we need $29 = ab$ with $1 \leq b/a \leq 2$.
- For the Oregon arrangement, we need $31 = 29 + 2 = ab$ with $1 \leq b/a \leq 2$.
- For the Wyoming arrangement, we need $27 = 29 - 2 = ab$ with a and b close to each other.
- For the remaining arrangements (long, short), we need 59 or 57 to factor as a product ab with a and b close to each other.

Characterization of Arrangements

A nice arrangement of n stars on the Union Jack exists if at least one of the following holds:

(i) For the long pattern, $2n - 1 = (2a + 1)(2b + 1)$ with $1 \leq (b + 1)/(2a + 1) \leq 2$.

(ii) For the short pattern, $2n + 1 = (2a - 1)(2b + 1)$ with $1 \leq (b + 1)/(2a + 1) \leq 2$.

(iii) For the alternate pattern, $n = a(2b - 1)$ with $1 \leq b/(2a) \leq 2$.

(iv) For the Wyoming pattern, $n - 2 = ab$ with $1 \leq (b + 1)/a \leq 2$.

(v) For the equal pattern, $n = ab$ with $1 \leq b/a \leq 2$.

(vi) For the Oregon pattern, $n + 2 = ab$ with $1 \leq b/a \leq 2$.

Notation

- We write $f(N) = O(g(N))$ if

$$|f(N)| \leq c|g(N)|$$

  for some constant $c > 0$ as $N \to \infty$.
- For a set of non-negative integers A, we call

$$\lim_{N \to \infty} \frac{\#\{n \leq N : n \in A\}}{N}$$

  the **asymptotic density** of A.

$\Omega(n)$

- Let $\Omega(n) = \sum_{\substack{p^a | n \\ a \geq 1}} 1$.
- $\Omega(20) = \Omega(4) + \Omega(5) = 2 + 1 = 3$
- One can show that $\frac{1}{N} \sum_{n \leq N} \Omega(n) = \log \log N + O(1)$.
- For $n \leq 10^{100}$, $\Omega(n) \approx 6$.

Sketch of the proof for $\Omega(n)$

Theorem (Mertens)

We have that

$$\sum_{p \le N} \frac{1}{p} = \log\log N + O(1).$$

Sketch of the proof for $\Omega(n)$

From Mertens' result, it follows that

$$\begin{aligned}
\frac{1}{N}\sum_{n\le N}\Omega(n) &= \frac{1}{N}\sum_{n\le N}\sum_{\substack{p^a\mid n\\ a\ge 1}} 1\\
&= \frac{1}{N}\sum_{\substack{p^a\le N\\ a\ge 1}}\sum_{\substack{n\le N\\ p^a\mid n}} 1\\
&= \frac{1}{N}\sum_{\substack{p^a\le N\\ a\ge 1}}\left(\frac{N}{p^a}+O(1)\right)\\
&= \sum_{p\le N}\frac{1}{p}+O\left(\sum_{\substack{p\text{ prime}\\ a\ge 2}}\frac{1}{p^a}\right) \quad = \log\log N + O(1).
\end{aligned}$$

Theorem (Hardy, Ramanujan)

For any $\epsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ n \leq N : |\Omega(n) - \log\log N| \leq \epsilon \log\log N \right\} = 1.$$

The Multiplication Table Problem

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

The Multiplication Table Problem

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2						12	14	16	18	20
3					15		21	24	27	30
4							28	32	36	40
5					25		35		45	50
6							42	48	54	60
7							49	56	63	70
8								64	72	80
9									81	90
10										100

Heuristic argument

- Let $A(N) = \#\{n \leq N : n = n_1 n_2, n_i \leq \sqrt{N}\}$
- Suppose n is in the multiplication table where the axis ranges from 1 to $\sqrt{N}$.
- $n = n_1 n_2 \Rightarrow \Omega(n) = \Omega(n_1) + \Omega(n_2) \approx 2 \log\log \sqrt{N}$
- This implies that $\Omega(n) \approx 2 \log\log N$, which is not very common.

  Note: $\log\log \sqrt{N} = \log\log N + \log\log 1/2 \approx \log\log N$.
- We expect $A(N)$ to be small.

A more complete argument

Suppose that n is an integer counted by $A(N)$. Then $n = n_1 n_2$ with $n_1, n_2 \leq \sqrt{N}$ and either

Case 1: $\Omega(n_1) < \frac{2}{3} \log\log N$ or

Case 2: $\Omega(n_2) \geq \Omega(n_1) \geq \frac{2}{3} \log\log N \Rightarrow \Omega(n) \geq \frac{4}{3} \log\log N$.

The number of integers $n \leq N$ counted by Case 1 is at most

$$\#\left\{n_1 \leq \sqrt{N} : \Omega(n_1) \leq \frac{2}{3} \log\log N\right\} \cdot \#\left\{n_2 \leq \sqrt{N}\right\}.$$

The number of integers $n \leq N$ counted by Case 2 is at most

$$\#\left\{n \leq N : \Omega(n) \geq \frac{4}{3} \log\log N\right\}.$$

A more complete argument

So

$$\frac{A(N)}{N} \leq \frac{1}{\sqrt{N}} \# \left\{ n_1 \leq \sqrt{N} : \Omega(n_1) \leq \frac{2}{3} \log \log N \right\} + \frac{1}{N} \# \left\{ n \leq N : \Omega(n) \geq \frac{4}{3} \log \log N \right\} \to 0$$

as $N \to \infty$.

Results on $A(N)$

Erdős (1960)

$$A(N) = O_\epsilon\left(\frac{N}{(\log N)^{\delta-\epsilon}}\right), \delta = 1 - \frac{1+\log\log 2}{\log 2} = 0.086071\ldots$$

Hall–Tenenbaum (1988)

$$A(N) = O\left(\frac{N}{(\log N)^{\delta}\sqrt{\log\log N}}\right)$$

Ford (2008)

$$A(N) \asymp \frac{N}{(\log N)^{\delta}(\log\log N)^{3/2}}$$

Equal Patterns

- Let $n \leq N$ be such that it admits an equal star arrangement.
- So $n = ab$ with $1 \leq \frac{b}{a} \leq 2$.
- $a \leq b \leq 2a \Rightarrow \sqrt{n} \leq b \leq 2\sqrt{n}$.
- In particular, $a, b \leq 2\sqrt{N}$, so n is counted by $A(4N)$.
- This implies that such n have asymptotic density 0.

Note: We can argue in a similar way for the remaining arrangements.

Theorem (Koukoulopoulos, T)

The set of non-negative integers allowing for a nice arrangement of n stars on the U.S. flag has asymptotic density zero.

Thank you!