MAA Metro NY
Problem of the Month

October 2026

Establish that the coefficient of x^k in the expansion of e^{e^x} is given by $\sum_{j=1}^{\infty} \frac{j^k}{j!k!}$ and deduce the exact value of $\sum_{j=1}^{\infty} \frac{j^5}{j!}$.

Computer or AI assisted/generated solutions will not be accepted. Please send your solution to maanyproblems@gmail.com, including your name (your given name first) and school affiliation if applicable. Solutions are due on October 31.

Solution by Henry Ricardo, Westchester Area Math Circle.

The Bell numbers B_n counts the total number of ways to partition a set of *n* distinct elements into non-empty, disjoint subsets. It is well known that the exponential generating function of the Bell numbers is

$$B(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = e^{e^x - 1}, \qquad (1)$$

so that $e\,B(x) = e^{e^x}$. Furthermore, the Bell numbers satisfy Dobiński's formula

$$B_n = \frac{1}{e} \sum_{j=0}^{\infty} \frac{j^n}{j!}. \qquad (2)$$

Substituting (2) in (1), we have

$$e^{e^x} = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{j^n}{j!n!} x^n = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \frac{j^n}{j!n!} x^n. \qquad (3)$$

From (3) we see that the coefficient of x^k in e^{e^x} is $\sum_{j=1}^{\infty} \frac{j^k}{j!k!}$. It follows from (2) that

$$\sum_{j=1}^{\infty} \frac{j^5}{j!} = e \cdot B_5 = 52e.$$