MAA Metro NY Problem of the Month

June 2026

Carefully sketch the region enclosed by

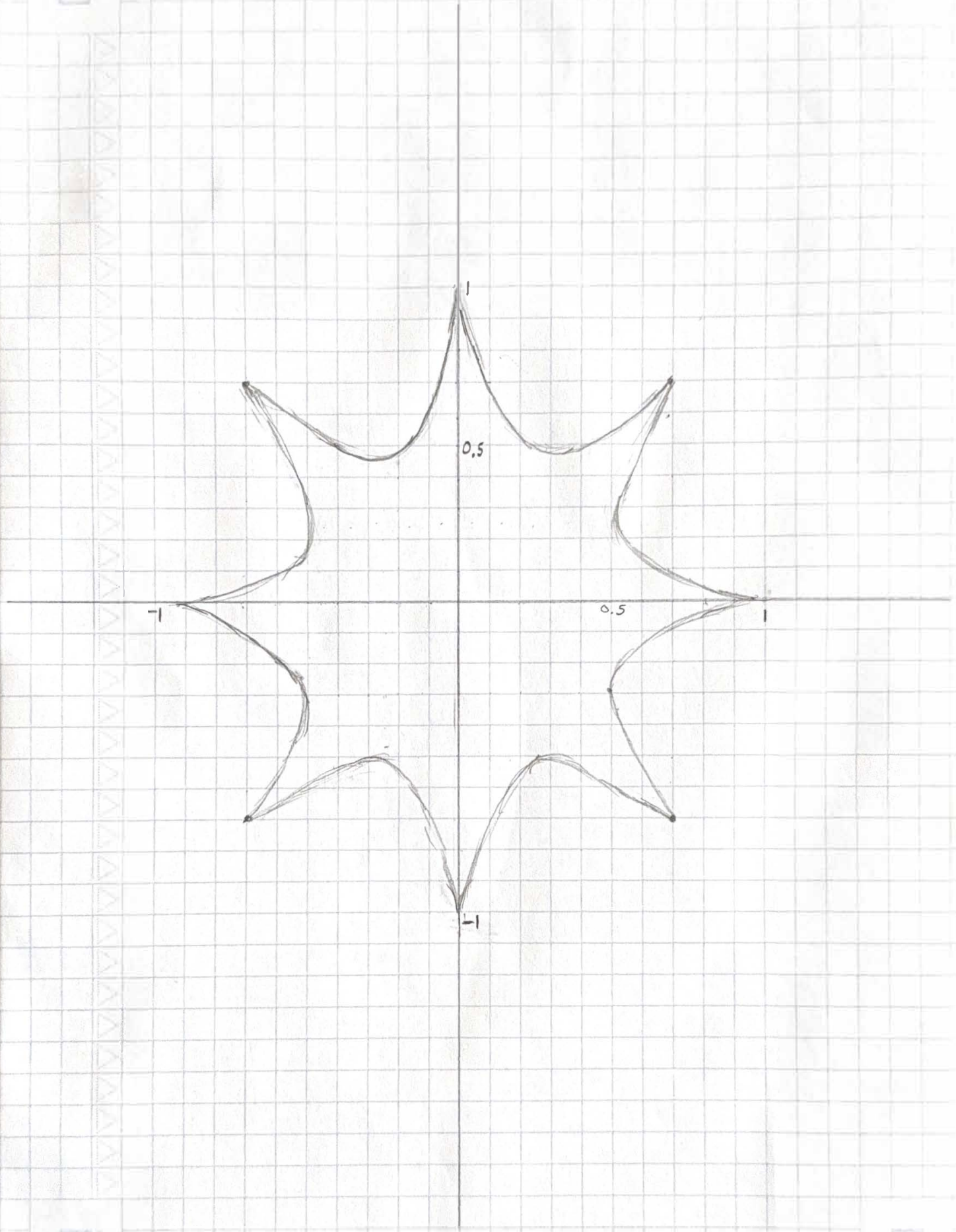
$$\frac{(2xy)^{2/3} + (x^2 - y^2)^{2/3}}{(x^2 + y^2)^{1/3}} = 1$$

and find the area enclosed by the curve.

Computer or AI assisted/generated solutions will not be accepted. Please send your solution to maanyproblems@gmail.com, including your name (your given name first) and school affiliation if applicable. Solutions are due on June 30.

We feature two solutions: Jacob Edelson of Hofstra University and Juergen Weith of Technical University of Applied Sciences Wuerzburg-Schweinfurt (THWS)

Jacob Edelson, Hofstra University



$$1 = \frac{(2r^2\sin(\theta)\cos(\theta))^{2/3} + (r^2(\cos^2(\theta) - \sin^2(\theta)))^{2/3}}{(r^2(\cos^2(\theta) + \sin^2(\theta)))^{1/3}}$$

$$1 = \frac{r^{4/3}(2\sin(\theta)\cos(\theta))^{2/3} + r^{4/3}(\cos(2\theta))^{2/3}}{r^{2/3}}$$

$$1 = r^{2/3}((\sin(2\theta))^{2/3} + (\cos(2\theta))^{2/3})$$

$$\frac{1}{r^{2/3}} = (\sin(2\theta))^{2/3} + (\cos(2\theta))^{2/3}$$

$$r^{2/3} = \frac{1}{(\sin(2\theta))^{2/3} + (\cos(2\theta))^{2/3}}$$

$$r = \frac{1}{((\sin(2\theta))^{2/3} + (\cos(2\theta))^{2/3})^{3/2}}$$

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2(\theta)\,d\theta$$

$$A = \frac{1}{2}\cdot 16\int_0^{\pi/8} \frac{1}{((\sin(2\theta))^{2/3} + (\cos(2\theta))^{2/3})^3}\,d\theta$$

$$= 8\int_0^{\pi/8} \frac{d\theta}{(\cos^{2/3}(2\theta)(+\tan^{2/3}(2\theta)+1))^3}$$

$$= 8\int_0^{\pi/8} \frac{d\theta}{\cos^2(2\theta)(+\tan^{2/3}(2\theta)+1)^3}$$

$$= 8\int_0^{\pi/8} \frac{\sec^2(2\theta)}{(+\tan^{2/3}(2\theta)+1)^3}\,d\theta$$

$u = \tan(2\theta) \quad du = 2\sec^2(2\theta)\,d\theta \Rightarrow \frac{du}{2} = \sec^2(2\theta)\,d\theta$

$$= 4\int_0^1 \frac{du}{(u^{2/3}+1)^3}$$

$w = u^{2/3} + 1 \qquad dw = \frac{2}{3}u^{-1/3}\,du$

$w - 1 = u^{2/3} \qquad \frac{3}{2}u^{1/3}\,dw = du$

$(w-1)^{3/2} = u$

$$\frac{3}{2}((w-1)^{3/2})^{1/3}\,dw = du$$

$$du = \frac{3}{2}\sqrt{w-1}\,dw$$

$$= 6\int_0^2 \frac{\sqrt{w-1}}{w^3}\,dw \longrightarrow$$

$$= 6\int_1^2 \frac{\sqrt{w-1}}{w^3}\,dw$$

$t = \sqrt{w-1}$ $\quad dt = \frac{1}{2\sqrt{w-1}}$

$dv = w^{-3}$ $\quad v = -\frac{1}{2w^2}$

$$= 6\left(-\frac{\sqrt{w-1}}{2w^2}\Big|_1^2 + \frac{1}{4}\int_1^2 \frac{dw}{w^2\sqrt{w-1}}\right)$$

$z = \sqrt{w-1}$ $\quad dz = \frac{dw}{2\sqrt{w-1}}$

$z^2 = w-1$ $\quad dz = \frac{dw}{2\sqrt{z^2+1-1}}$

$z^2+1 = w$

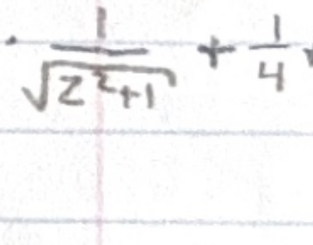
$w^2 = (z^2+1)^2$ $\quad 2z\,dz = dw$

$$\frac{1}{2}\int_0^1 \frac{z}{(z^2+1)^2 \cdot z}\,dz$$

$$\frac{1}{2}\int_0^1 \frac{dz}{(z^2+1)^2}$$

$\tan\phi = z$

$dz = \sec^2\phi\,d\phi$



$$\frac{1}{2}\int \frac{\sec^2\phi}{(\tan^2\phi+1)^2}\,d\phi = \frac{1}{2}\int \frac{\sec^2\phi}{\sec^4\phi}\,d\phi$$

$$= \frac{1}{2}\int \cos^2\phi\,d\phi$$

$$= \frac{1}{2}\int \frac{1}{2}\cos(2\phi) + \frac{1}{2}\,d\phi$$

$$= \frac{1}{2}\left(\frac{1}{4}\sin(2\phi) + \frac{\phi}{2}\right)$$

$$= \frac{1}{8}\sin(2\phi) + \frac{\phi}{4}$$

$$= \frac{1}{8}(2\sin(\phi)\cos(\phi)) + \frac{\phi}{4}$$

$$= \frac{1}{4}\sin(\phi)\cos(\phi) + \frac{\phi}{4}$$

$$= \frac{1}{4}\cdot\frac{z}{\sqrt{z^2+1}}\cdot\frac{1}{\sqrt{z^2+1}} + \frac{1}{4}\tan^{-1}(z)$$

$$= 6\left(-\frac{\sqrt{w-1}}{2w^2}\Big|_1^2 + \left[\frac{z}{4(z^2+1)} + \frac{1}{4}\tan^{-1}(z)\right]\Big|_0^1\right)$$

$$= 6\left(-\frac{\sqrt{2-1}}{2(2)^2} + \cancel{\frac{\sqrt{1-1}}{2(1)^2}}^{\,0} + \frac{1}{4(1^2+1)} + \frac{1}{4}\tan^{-1}(1) - \cancel{\frac{0}{4(0-1)}}^{\,0} + \cancel{\frac{1}{4}\tan^{-1}(0)}^{\,0}\right)$$

$$= 6\left(-\frac{1}{8} + \frac{1}{8} + \frac{1}{4}\left(\frac{\pi}{4}\right)\right)$$

$$= \frac{6\pi}{16} = \frac{3\pi}{8}$$

$$\boxed{\frac{3\pi}{8}}$$

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MAA Metro NY Problem of the Month May 2026

June 29, 2026

Problem Statement

Carefully sketch the region enclosed by

$$\frac{(2xy)^{2/3} + (x^2 - y^2)^{2/3}}{(x^2 + y^2)^{1/3}} = 1 \tag{1}$$

and find the area A enclosed by the curve.

My Solution

The curve is a fourfold symmetric closed algebraic curve.
The area is given by $A = \frac{3\pi}{8} = 1.1780972451\ldots$
This result was analytically derived and verified by Monte Carlo simulation.

My Proof

Problems like this can often be solved more easily by using polar coordinates. This is due to the fact that the area A can be represented in a simple manner by the integral

$$A = \frac{1}{2}\int_0^{2\pi} r^2(\theta)\,\mathrm{d}\theta \tag{2}$$

where r and θ are the radial and angular coordinates respectively.

1. **Coordinate Transformation**
   We start with

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta. \end{aligned}$$

   For the formula given in the problem statement, we require:

$$\begin{aligned} 2xy &= r^2\sin(2\theta) \\ x^2 - y^2 &= r^2\cos(2\theta) \\ x^2 + y^2 &= r^2. \end{aligned}$$

Substituting these expressions yields

$$\frac{(r^2\sin 2\theta)^{2/3}+(r^2\cos 2\theta)^{2/3}}{(r^2)^{1/3}}=1$$

$$r^{2/3}\left(|\sin 2\theta|^{2/3}+|\cos 2\theta|^{2/3}\right)=1$$

and from this:

$$r=\left(|\sin 2\theta|^{2/3}+|\cos 2\theta|^{2/3}\right)^{-3/2}.$$

2. Performing the Integration

The next step is to perform the integration (2):

$$A=\frac{1}{2}\int_0^{2\pi} r^2(\theta)\,\mathrm{d}\theta=\frac{1}{2}\int_0^{2\pi}\left(|\sin 2\theta|^{2/3}+|\cos 2\theta|^{2/3}\right)^{-3}\mathrm{d}\theta.$$

The integrand has a periodicity of $\pi/4$, so our integral reduces to

$$A=4\int_0^{\pi/4}\left(|\sin 2\theta|^{2/3}+|\cos 2\theta|^{2/3}\right)^{-3}\mathrm{d}\theta.$$

Let $u=2\theta$, then:

$$A=2\int_0^{\pi/2}\frac{\mathrm{d}u}{\left(\sin^{2/3}u+\cos^{2/3}u\right)^3}.$$

To proceed towards the solution, we substitute

$$t=\tan u.$$

Since

$$\sin u=\frac{t}{\sqrt{1+t^2}},\qquad \cos u=\frac{1}{\sqrt{1+t^2}},\qquad \mathrm{d}u=\frac{\mathrm{d}t}{1+t^2}$$

we obtain

$$A=2\int_0^{\infty}\frac{\mathrm{d}t}{(1+t^{2/3})^3}.$$

Using the following substitutions:

$$s=t^{2/3},\qquad t=s^{3/2},\qquad \mathrm{d}t=\frac{3}{2}s^{1/2}\,\mathrm{d}s$$

we now have

$$A=3\int_0^{\infty}\frac{s^{1/2}}{(1+s)^3}\,\mathrm{d}s. \tag{3}$$

3. Introducing the Euler Integral of the First Kind

Equation (3) is related to the Euler integral:

$$\int_0^{\infty}\frac{s^{a-1}}{(1+s)^{a+b}}\,\mathrm{d}s=B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

For our case

$$a = b = \frac{3}{2}.$$

We thus have

$$A = 3B\left(\frac{3}{2}, \frac{3}{2}\right) = 3\frac{\Gamma(3/2)^2}{\Gamma(3)}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}, \quad \Gamma(3) = 2$$

4. Final Result

$$A = 3 \cdot \frac{\pi/4}{2} = \frac{3\pi}{8} \tag{4}$$

5. Performing a Monte Carlo Area Simulation

To ensure the correctness of the theoretical result (4), a Monte Carlo analysis was performed. A total of $N = 10^7$ (in x and y direction) uniformly distributed points were generated within a square of area 4 (dimensions 2×2), yielding N_i points inside the region enclosed by the curve (1). The approximation of the area A is given by $A = 4\frac{N_i}{N}$. The graphical representation of the simulation result is depicted in Figure 1. The calculated value confirms the theoretical analysis: It is within the margin of the expected simulation accuracy (rel. error=2.93e-04).

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Monte Carlo Simulation Results
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Total points (N):   10000000

Estimated Area:     1.17844240
Exact Value:        1.17809725

Absolute Error:     0.00034515
Relative Error:     2.9298e-04
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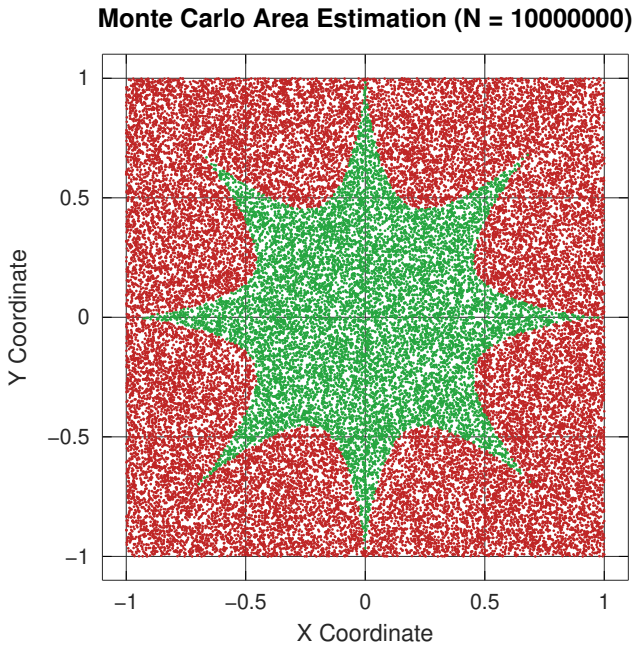


Figure 1: Monte Carlo simulation for area estimation. The green points fall inside the curve boundary, while the red points lie outside.